for some scalar \( p \). Equations (6.3) and (6.4) form a set of necessary conditions for a solution of the constrained optimization problem. The necessary conditions for optimality, (6.3) and (6.4), can be generated as the solution to an unconstrained optimization problem with the following cost function:

\[
J_a(x, p) = J(x) + pc(x).
\]  

(6.5)

Taking the gradient of \( J_a \) with respect to \( x \) yields (6.4), and taking the derivative of \( J_a \) with respect to \( p \) yields (6.3). The parameter \( p \) is called a Lagrange multiplier. The procedure of solving the constrained optimization problem for \( J \) by solving the unconstrained optimization problem for \( J_a \) is called the method of Lagrange multipliers. This method is also applicable to the optimal control problem, which involves the constrained minimization of a functional.

**Example 6.2**

We are given the cost function and the constraint

\[
J(x, y) = x^2 + y^2;
\]

\[
c(x, y) = 2x + y + 4 = 0.
\]

The augmented cost function is

\[
J_a(x, y, p) = x^2 + y^2 + p(2x + y + 4).
\]

The increment of the augmented cost function is:

\[
\Delta J_a(x, y, p, \delta x, \delta y, \delta p) = J_a(x + \delta x, y + \delta y, p + \delta p) - J_a(x, y, p)
\]

\[
= (x + \delta x)^2 + (y + \delta y)^2
\]

\[
+ (p + \delta p)(2(x + \delta x) + (y + \delta y) + 4)
\]

\[
- x^2 - y^2 - p(2x + y + 4)
\]

\[
= (2x + 2p)\delta x + (2y + p)\delta y + (2x + y + 4)\delta p
\]

\[
+ \delta x^2 + \delta y^2 + 2\delta p\delta x + \delta p\delta y.
\]

A necessary condition for optimality is that the variation of the augmented cost function, which consists of the linear terms in the increment, is zero for all \( \delta x, \delta y, \) and \( \delta p \):

\[
\delta J_a(x, y, p, \delta x, \delta y, \delta p) = (2x + 2p)\delta x + (2y + p)\delta y + (2x + y + 4)\delta p = 0.
\]

Therefore, the coefficients that multiply \( \delta x, \delta y, \) and \( \delta p \) must all equal zero:

\[
\frac{\delta J_a(x, y, p)}{\delta x} = 2x + 2p = 0;
\]

\[
\frac{\delta J_a(x, y, p)}{\delta y} = 2y + p = 0;
\]

\[
\frac{\delta J_a(x, y, p)}{\delta p} = 2x + y + 4 = 0.
\]