Note that state and control weightings are incorporated into the output equation, which is scaled to yield a unity weight on the control. The Hamiltonian system is

\[
\begin{bmatrix}
    \dot{x}(t) \\
    \dot{p}(t)
\end{bmatrix} =
\begin{bmatrix}
    1 & -1 + 4\gamma^2 \\
    -100 & -1
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    p(t)
\end{bmatrix},
\]

which has the state-transition matrix

\[
\Phi(t) = e^{\mathbf{X}_t} = \mathcal{L}^{-1}((s\mathbf{I} - \mathbf{X}_s)^{-1})
\]

\[
= \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 101 + 400\gamma^2} \begin{bmatrix}
    s + 1 & -1 + 4\gamma^2 \\
    -100 & s - 1
\end{bmatrix}\right\}.
\]

A saddle point of this differential game exists (and an \(\mathcal{H}_\infty\) full information suboptimal controller exists) for a given performance bound provided \(\Phi_{22}(t_f - t)\) has an inverse throughout the time interval; that is, the scalar \(\Phi_{22}(t_f - t) \neq 0\). This element of the state transition matrix can be computed:

\[
\Phi_{22}(t) = \mathcal{L}^{-1}\left\{\frac{s - 1}{s^2 - 101 + 400\gamma^2}\right\} = \begin{cases}
    \frac{a - 1}{2a} e^{at} + \frac{a + 1}{2a} e^{-at} & \gamma > \sqrt{\frac{400}{101}} \\
    1 - t & \gamma = \sqrt{\frac{400}{101}} \\
    \frac{\sqrt{\omega^2 + 1}}{\omega^2} \sin(\omega t + \theta) & \gamma < \sqrt{\frac{400}{101}}
\end{cases}
\]

where \(a = \sqrt{101 - 400\gamma^2}\), \(\omega = \sqrt{-101 + 400\gamma^2}\), and \(\theta = -\tan^{-1}(\omega)\). Note that as \(\gamma\) gets smaller, \(a\) gets smaller, \(\omega\) gets bigger, and \(\theta\) gets more negative. When \(\gamma < \sqrt{400/101}\), the eigenvalues of the Hamiltonian matrix are imaginary, and the gain does not exist for long time intervals, since \(\Phi_{22}(t)\) periodically crosses zero. Further, the fact that the feedback gain does not exist implies that there is no solution of the differential game (and no solution of the \(\mathcal{H}_\infty\) suboptimal control problem) for the given bound. Less obvious from these expressions is that the time interval over which a feedback gain exists becomes monotonically larger as \(\gamma\) is increased, reaching infinity for \(\gamma > 2\).

For the bound \(\gamma = 2.03\) (\(a = 2\)), and the final time \(t_f = 3\), the \(\mathcal{H}_\infty\) suboptimal feedback gain is

\[
K(t) = \frac{100e^{2(t-\tau)} - 100e^{-2(t-\tau)}}{e^{2(t-\tau)} + 3e^{-2(t-\tau)}},
\]

which is plotted in Figure 9.2. Note that the gain exhibits a transient and then approaches a steady-state value far from the final time. In situations where the time interval is long compared to the settling time of this transient, it may be reasonable to use only the steady-state gain.

An \(\mathcal{H}_\infty\) suboptimal controller fails to exist over long time intervals, in the above example, when the Hamiltonian has purely imaginary eigenvalues. This result is a general property of \(\mathcal{H}_\infty\) suboptimal control:

The feedback gain exists for arbitrary time intervals, and there is a solution to the \(\mathcal{H}_\infty\) suboptimal control problem, only if the Hamiltonian matrix has no purely imaginary eigenvalues.