An estimator that is stable and bounds the $\infty$-norm of the closed-loop transfer function (9.33) exists if and only if there exists a positive semidefinite solution of the algebraic Riccati equation

$$
\mathbf{A} \mathbf{Q} + \mathbf{Q} \mathbf{A}^T - \mathbf{Q} \left( \mathbf{C}_m^T \mathbf{C}_m - \gamma^{-2} \mathbf{C}_p^T \mathbf{C}_p \right) \mathbf{Q} + \mathbf{B}_w \mathbf{B}_w^T = \mathbf{0},
$$

such that the matrix

$$
\mathbf{A} - \mathbf{Q} \left( \mathbf{C}_m^T \mathbf{C}_m - \gamma^{-2} \mathbf{C}_p^T \mathbf{C}_p \right)
$$

is stable.

The steady-state Riccati solution can be found from the eigensolution of the Hamiltonian:

$$
\mathbf{Q} = \mathbf{\Psi}_n \left( \mathbf{\Psi}_n \right)^{-1},
$$

where

$$
\begin{bmatrix}
\mathbf{\Psi}_{1n} \\
\mathbf{\Psi}_{2n}
\end{bmatrix}
= \begin{bmatrix}
\gamma_{12} \\
\gamma_{22}
\end{bmatrix}
$$

is a matrix whose columns are the eigenvectors of the Hamiltonian associated with the unstable eigenvalues. As observed previously, the $\mathcal{H}_\infty$ optimal estimator can be approximated to an arbitrary degree of closeness by iteration of the bound and testing for the existence of suboptimal estimators.

**Example 9.4**

An ac motor is described by the following state equation:

$$
\dot{x}(t) = -x(t) + 100u(t) + 100w(t),
$$

where $x(t)$ is the rotational velocity of the shaft, $u(t)$ is the nominal input of 10 volts rms, and $w(t)$ is the error between the actual applied voltage and the nominal value. This error is assumed to be caused by modulation of the ac envelope due to harmonics on the power line. This modulation is assumed to be less than 1%, implying the disturbance input can be bounded:

$$
|w(t)| \leq 0.1 \text{ volts rms}.
$$

An $\mathcal{H}_\infty$ estimator is used to estimate the true rotational velocity of the shaft from the noisy tachometer measurements:

$$
m(t) = x(t) + v(t),
$$

where $v(t)$ is the measurement error. This error consists of a possible dc bias and sinusoidal interference, and is bounded:

$$
|v(t)| \leq 1 \text{ volts rms}.
$$

The output of interest is simply the state.