where the ordinary transpose is used for $S$, since this matrix is real. The maximum of the norm (A3.4) is

$$\max_{l=1}^{\infty} \|Mx_l\| = \max_{l=1}^{\infty} \sqrt{x_l^T V^T S V x_l} = \max_{l=1}^{\infty} \sqrt{x_l^T S^T S x_l}.$$  

Note that maximization over $x_l = V z_l$ is equivalent to maximizing over $x$ since $V$ is invertible and preserves the norm (1 in this case). Expanding the norm yields

$$\max_{l=1}^{\infty} \|Mx_l\| = \max_{l=1}^{\infty} \sqrt{x_l^T S^T S x_l} = \max_{l=1}^{\infty} \sqrt{\sigma_1^2 |x_l|^2 + \sigma_2^2 |x_{2l}|^2 + \cdots + \sigma_{n_u}^2 |x_{n_u}|^2}.$$  

This expression is maximized, given the constraint $\|z_l\| = 1$, when $x$ is concentrated at the largest singular value; that is $|x_l| = [1 \ 0 \ \cdots \ 0]^T$. The maximum gain is then

$$\max_{l=1}^{\infty} \|Mx_l\| = \sqrt{\sigma_1^2} [1]^2 + \sigma_2^2 [0]^2 + \cdots + \sigma_{n_u}^2 [0]^2 = n_u = \hat{\sigma}.$$  

**THEOREM:** The minimum gain of a matrix is given by the smallest singular value:

$$\min_{l=1}^{\infty} \|Mx_l\| = \sigma_{n_u} = \sigma = \begin{cases} \sigma_{p} & n_y \geq n_u \\ 0 & n_y < n_u \end{cases}. \tag{A3.6}$$

**PROOF:** Substituting the minimum operator for the maximum operator in (A3.5), the minimum gain is seen to occur when $x$ is concentrated at the smallest singular value; that is, $|x_l| = [0 \ \cdots \ 0 1]^T$. The minimum gain in this case is

$$\min_{l=1}^{\infty} \|Mx_l\| = \sigma_{n_u} = \sigma = \begin{cases} \sigma_{p} & n_y \geq n_u \\ 0 & n_y < n_u \end{cases}.$$  

**THEOREM:** The maximum singular value of a square matrix's inverse equals the inverse of the matrix's minimum singular value:

$$\hat{\sigma}(M^{-1}) = \frac{1}{\hat{\sigma}(M)}. \tag{A3.7}$$

**PROOF:** The SVD of $M^{-1}$ is given (provided all inverses exist):

$$M^{-1} = (US^{-1}V^*)^{-1} = VS^{-1}U^T = \sum_{i=1}^{p} \sigma_i^{-1} V_i U_i^T,$$

where $\sigma_i$, $U_i$, and $V_i$ are the singular values, left singular vectors, and right singular vectors of $M$, respectively. Note that the inverse of a unitary matrix exists and equals the conjugate transpose of the matrix. The matrix $S$ is a diagonal matrix with no zeros on the diagonal, since $M$ is full rank. Therefore, the inverse of $S$ exists and is a diagonal matrix with the inverses of the singular values on the diagonal. The maximum singular value of $M^{-1}$ is then $1/\hat{\sigma}$.  

$\hat{\sigma}$.