The maximum singular value at a specific frequency is bounded by the supremum over all frequencies:

\[
\|y\|_2 \leq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\sigma}(j\omega) \|u(j\omega)\|_2^2 d\omega} \leq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \sup_{\omega} \tilde{\sigma}(j\omega) \right\} \|u(j\omega)\|_2^2 d\omega} \\
= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G\|_\infty^2 \|u(j\omega)\|_2^2 d\omega} = \|G\|_\infty \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|u(j\omega)\|_2^2 d\omega} = \|G\|_\infty \|u\|_2.
\]

The bound of the preceding proof is nearly achieved for some input. In fact, the \(\infty\)-norm is given by the supremum:

**Theorem:** \(\|G\|_\infty = \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2}. \) (A9.3)

**Proof:** Using (A9.2),

\[
\|G\|_\infty \geq \frac{\|Gu\|_2}{\|u\|_2} \quad \forall u \neq 0,
\]

which implies

\[
\|G\|_\infty \geq \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2}.
\]

The input direction and frequency that yield the maximum in the \(\infty\)-norm are defined as follows:

\[
\|G\|_\infty = \sup_{u \neq 0} \tilde{\sigma}(G(j\omega)) = \tilde{\sigma}(G(j\omega_0)) = \max_{u \neq 0} \frac{\|G(j\omega_0)u\|_2}{\|u\|_2} \frac{\|G(j\omega_0)u_0\|_2}{\|u_0\|_2}
\]

where \(u_0\) is a unit vector. Note that \(u_0\) is the right singular vector associated with the largest singular value. For the input

\[
u(j\omega) = u_0 \delta(\omega - \omega_0), \quad \|v\|_\infty
\]

\[
\frac{\|Gu\|_2}{\|u\|_2} = \frac{\|G(j\omega_0)u_0 \delta(\omega - \omega_0)\|_2}{\|u_0\|_2} \frac{\|G(j\omega_0)u_0 \delta(\omega - \omega_0)\|_2}{\|u_0\|_2} = \frac{\|G(j\omega_0)u_0\|_2}{\|u_0\|_2} \frac{\|u_0\|_2}{\|u_0\|_2} = \|G\|_\infty.
\]

The above proof is not rigorous, since the supremum has been treated as a maximum. The supremum says only that the limit can be approached arbitrarily closely, not that the value can be achieved. The theorem remains unchanged when using the actual supremum, but the proof must include a limiting argument that has been omitted.

In addition to the ordinary properties of norms, the infinity norm has the following property: