Using (A11.3),
\[
\tilde{g}(t, \tau) = B^T(t_f - t) \Phi^T(t_f - \tau, t_f - t) C^T(t_f - \tau) + D^T(t_f - t) \delta(t - \tau) = g^T(t_f - \tau, t_f - t).
\]

**Theorem:** The system 2-norms of the original and the adjoint systems are related:

\[
\|\tilde{g}\|_{2, [t_0, t_f]} = \|\tilde{g}\|_{2, [0, t_f - t_0]}.
\]  
(A11.5)

**Proof:** The system 2-norm of the adjoint system is

\[
\|\tilde{g}\|_{2, [0, t_f - t_0]}^2 = \frac{1}{t_f - t_0} \int_{t_0}^{t_f-t_0} \int_{t_0}^{t_f-t_0} \text{trace}(g^T(t_f - \tau, t_f - t) g(t_f - \tau, t_f - t)) d\tau dt
\]

\[
= \frac{1}{t_f - t_0} \int_{t_0}^{t_f-t_0} \int_{t_0}^{t_f-t_0} \text{trace}(g^T(\tau, \tau) g(\tau, \tau)) d\tau dt = \|g\|_{2, [t_f, t_0]}^2,
\]

where the second integral expression is arrived at by using the fact that the trace is invariant under cyclic permutations (A2.3), changing the order of integration, and performing the change of variables \( t \rightarrow t_f - \tau; \tau \rightarrow t_f - t \).

**Theorem:** The system \(\infty\)-norms of the original and the adjoint systems are related:

\[
\|\tilde{g}\|_{\infty, [t_0, t_f]} = \|\tilde{g}\|_{\infty, [0, t_f - t_0]}.
\]  
(A11.6)

**Proof:** The \(\infty\)-norm of the adjoint system is

\[
\|\tilde{g}\|_{\infty, [0, t_f - t_0]}^2 = \sup_{i \leq 1} \int_{t_0}^{t_f-t_0} \int_{t_0}^{t_f-t_0} \tilde{g}(\tau_1) g^T(t_f, \tau_1) \tilde{g}(t_f, \tau_2) g(t_f, \tau_2) d\tau_1 d\tau_2 dt.
\]

Substituting for the impulse response of the adjoint system using (A11.4) and performing the change of variables \( t \rightarrow t_f - \tau_1, \tau_1 \rightarrow t_f - \tau_1, \) and \( \tau_2 \rightarrow t_f - \tau_2 \) yields

\[
\|\tilde{g}\|_{\infty, [0, t_f - t_0]}^2 = \sup_{i \leq 1} \int_{t_0}^{t_f-t_0} \int_{t_0}^{t_f-t_0} \tilde{g}(\tau_1) g(t_f, \tau_1) g(t_f, \tau_2) g(t_f, \tau_2) d\tau_1 d\tau_2 dt
\]

\[
= \|g\|_{\infty, [t_f, t_0]}^2.
\]

The theorems (A11.5) and (A11.6) are also valid in the infinite-time case.

### A12 The Kalman Filter Innovations

The innovations (or residuals) of the Kalman filter are the errors between the measurement and the predicted measurement:

\[
\tilde{m}(t) = m(t) - C_m \hat{x}(t).
\]

**Theorem:** The innovations form a white noise random process with spectral density equal to the measurement noise spectral density \( S_m \).