\[
R_{[u_1]}(\tau) = C_{[u_2]}(\tau) = \begin{bmatrix}
4e^{-|\tau|} & 0 \\
0 & e^{-0.5|\tau|}
\end{bmatrix}.
\]

This correlation function is shown in Figure 3.3b. The fact that the off-diagonal terms in this correlation function are zero indicates that there is no correlation between the pitch and roll positions of the stick. A correlation time can be defined for each of these inputs:

\[
\tau_e^{(1)} = \frac{1}{4} \int_0^\infty 4e^{-\tau}d\tau = 1 \text{ sec}; \quad \tau_e^{(2)} = \frac{1}{4} \int_0^\infty e^{-0.5\tau}d\tau = 2 \text{ sec}.
\]

Note that these correlation times are a rough measure of the time it takes the sample function (see Figure 3.3a) to change appreciably; that is, the pitch stick position changes significantly about every second, while the roll stick position changes about every two seconds. Further, the component with the longer correlation time is also the component with the broader correlation function.

### 3.1.4 The Spectral Density

The frequency content of a stationary random process is described by the spectral density (often called the power spectral density):

\[
S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega \tau}d\tau.
\]

The diagonal elements of the spectral density are strictly real due to the symmetry of the correlation function (3.3) and, therefore, contain only magnitude information on the frequency content of the elements of the random process. This lack of phase information is inherent in the definition of stationarity. For example, consider a random process that is a sinusoid at a specified frequency, but with a random amplitude and phase. This sinusoid is only stationary if the phase angle is uniformly distributed between 0 and 2\(\pi\). If not, the preferred phase angle yields an increased probability of zero crossings at a particular time, and in turn a decrease in the mean at those times. The off-diagonal elements of the spectral density may be complex, in general. The nonzero argument of these elements indicates a relative phase between the two corresponding signals in the random process. The existence of a relative phase angle between signals in a random process does not violate the tenets of stationarity.

#### Example 3.3

The spectral density of the random signal in Example 3.3 is

\[
S_x(\omega) = \int_{-\infty}^{\infty} 0.20 \cos(0.5\tau)e^{-j\omega \tau}d\tau = 0.20 \delta(\omega - 0.5) + 0.20 \delta(\omega + 0.5).
\]

This spectral density (see Figure 3.4) shows that the entire frequency content of the waves is at a single frequency. This fact is quite reasonable (for the mathematical problem posed) since all of the sample functions are at a single frequency but, this is certainly an idealization of a true wave spectrum.