input to the system starts at time zero and can be modeled as the product of a unit-step function with a stationary input. This unit-step function can be expected to generate transients that add a time-varying component to the moments of the state. By analogy with deterministic results, these transients are expected to decay to zero with time for stable linear systems. The state is then approximately stationary far from the initial time.

A random process is termed asymptotically stationary in the wide sense if its mean, correlation matrix, and covariance matrix approach constants, and its correlation and covariance functions approach functions of the time difference between samples, as time increases to infinity. The state and output of the system of (3.4), with a random initial condition and subject to the stationary white noise input, are asymptotically stationary in the wide sense provided that the system is stable. This fact is demonstrated by noting that the mean of the state (and also the output) is zero since the means of the initial condition and the input are both zero. The steady-state correlation matrix of the state is obtained by taking the limit of (3.9):

$$\mathbf{\Sigma}_x(\infty) = \lim_{t \to \infty} \left\{ e^{\mathbf{A}_t} \mathbf{\Sigma}_x(0) e^{\mathbf{A}_t^T} + \int_0^t e^{\mathbf{A}_\gamma} \mathbf{B} \mathbf{S}_w \mathbf{B}^T e^{\mathbf{A}_\gamma^T} d\gamma \right\}.$$  

For a stable system, the matrix exponentials in this expression approach zero as time goes to infinity:

$$\mathbf{\Sigma}_x(\infty) = \int_0^\infty e^{\mathbf{A}_\gamma} \mathbf{B} \mathbf{S}_w \mathbf{B}^T e^{\mathbf{A}_\gamma^T} d\gamma.$$  

The integrand in this expression is a matrix whose elements are linear combinations of decaying exponentials, guaranteeing the existence of this improper integral. The output correlation matrix is obtained by substituting (3.12) into (3.8):

$$\mathbf{\Sigma}_y(\infty) = \int_0^\infty \mathbf{C} e^{\mathbf{A}_\gamma} \mathbf{B} \mathbf{S}_w \mathbf{B}^T e^{\mathbf{A}_\gamma^T} \mathbf{C}^T d\gamma = \int_0^\infty \mathbf{g}(\gamma) \mathbf{S}_w \mathbf{g}^T(\gamma) d\gamma$$

where \( \mathbf{g}(\gamma) \) is the impulse response matrix of the state model (3.4).

The steady-state correlation matrix of the state can also be found from the matrix differential equation (3.11) provided that the system is time-invariant and stable. In steady-state, the derivative of the correlation matrix equals zero:

$$\mathbf{A} \mathbf{\Sigma}_x(\infty) + \mathbf{\Sigma}_x(\infty) \mathbf{A}^T + \mathbf{B} \mathbf{S}_w \mathbf{B}^T = 0.$$  

This equation is known as a Lyapunov equation, and can be solved to yield the steady-state value of the state correlation matrix. A number of numerical techniques have been developed for the solution of Lyapunov equations, and the interested reader is referred to [2].