A STATE MODEL FOR THE AIR-FUEL RATIO

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A state model for the air-fuel ratio of a spark ignition engine is derived. This model is presented to provide a basic understanding of the dynamics which govern the air-fuel ratio. The actual process which generates the air-fuel ratio is very complex, and the derived model provides only an approximation to this process. The following explicit assumptions are made in deriving this model:

1. The air within the intake manifold acts as an ideal gas;
2. The rate of change of the temperature of the air within the intake manifold is small;
3. The individual cylinder events are ignored, i. e., only average flows are considered.

The resulting state model is appropriate for initial analysis and design of feedback systems for controlling the air-fuel ratio in a spark ignition engine.

Some basic results for ideal gases are presented in Section 1. These results are used in the derivation of the air flow dynamics. A state model for the air flow is developed in Section 2, and a state model for the fuel flow is given in Section 3. Section 4 presents the entire nonlinear state model for the air-fuel ratio. A table of parameter values (along with uncertainty information) for the 1992 Ford 4.6 liter-2 valve Modular V8 engine is also given in this section. Modifications to the basic models are presented in the Appendix. These modifications are presented to provide a better match between the models presented and the software used in test cell 13.

1. Ideal Gas Equations

The air within the intake manifold is modeled as an ideal gas and is, therefore, described by the ideal gas law:

\[ p_i V_i = m_i RT \]  \hspace{1cm} (1)

where \( p_i \) is the pressure in the intake manifold, \( V_i \) is the volume of the intake manifold, \( m_i \) is the mass of air within the intake manifold, \( R \) is the gas constant for air, and \( T \) is the temperature of the air within the intake manifold.\(^1\) This law can also be written in terms of the density of air within the intake manifold \( \rho_i \):

\[ \rho_i = \frac{m_i}{V_i} = \frac{p_i}{RT} . \]  \hspace{1cm} (2)

Further, the ideal gas law can be solved for the mass:

\(^1\) The ideal gas equation is usually given with \( n \) (the number of moles in the sample) in place of \( m \) (the mass of the sample). These two quantities are proportional to each other, and the constant of proportionality can be absorbed into the gas constant.
\[ m_i = \frac{p_i V_i}{RT}. \] (3)

Assuming that the time derivative of the temperature is small and can be ignored, the ideal gas law yields an expression for the time derivative of the mass rate:

\[ \frac{dm_i}{dt} = \frac{dp_i}{dt} \frac{V_i}{RT}. \] (4)

These relationships will be used below in the derivation of the air flow model.

2. The Air Flow Model

The rate of change of the mass of air within the intake manifold equals the mass entering through the throttle minus the mass exiting into the cylinders:

\[ \frac{dm_i}{dt} = \dot{m}_{th} - \sum_{cyl} \dot{m}_{cyl} \] (5)

where \( \dot{m}_{th} \) is the mass flow rate through the throttle and \( \dot{m}_{cyl} \) is the mass flow rate into each cylinder (the sum yields the total mass rate into the cylinders). The total mass rate into the cylinders can be given:

\[ \sum_{cyl} \dot{m}_{cyl} = \frac{\eta \rho_d V_d N}{2} \] (6)

where \( \eta \) is the volumetric efficiency of the engine, \( V_d \) is the displacement volume of the engine, and \( N \) is the rotation rate of the engine in revolutions per second. Equation (6) merely states that the mass rate equals the mass in a charge (the displacement volume times the volumetric efficiency times the density in the intake manifold) times the number of intake cycles per second (\( N/2 \)). The volumetric efficiency is a dimensionless correction that results because the cylinder does not get fully charged due to dynamic losses in the intake manifold, ports, and valve system. Residual exhaust gases within the cylinder and backflow of exhaust gases into the intake port also play a significant role in decreasing the volumetric efficiency. Substituting (6) into (5):

\[ \frac{dm_i}{dt} = \dot{m}_{th} - \frac{\eta \rho_d V_d N}{2}. \] (7)

Substituting for the derivative of the air mass within the intake manifold using (4) yields:

\[ \frac{dp_i}{dt} = \frac{\dot{m}_{th} RT}{V_i} - \frac{\eta \rho_d V_d NRT}{2V_i} \] (8)

Finally, substituting for the density in (8) using (2) yields a differential equation for the pressure within the intake manifold:
\[ \frac{dp_i}{dt} = -\frac{\eta V_a N}{2V_i} p_i + \frac{RT}{V_i} m_{th}. \]  

(9a)

This first order differential equation is in state equation form where the state is the intake manifold pressure, and the input is the mass flow rate of air through the throttle.

The air mass flow rate into the cylinders can be written in terms of the intake manifold pressure, by substituting for the density in (6) using (2):

\[ \sum_{cyl} \dot{m}_{cyl} = \frac{\eta V_a N}{2RT} p_i. \]  

(9b)

Equations (9a) and (9b) form a state model for the air flow where (9b) is the output equation. The coefficients in this state model are time varying since both \(N\) and \(T\) change with time. In addition, the state model is nonlinear since the volumetric efficiency of the engine is a nonlinear function of the pressure within the intake manifold.

3. The Fuel Flow Model

A state model for the fuel flowing into the cylinders is presented. This model is developed by assuming that only a portion of the fuel injected into the port vaporizes and directly enters the cylinder. The remaining fuel forms a “puddle” on the walls of the port and intake valve. The fuel in this puddle then slowly vaporizes and enters the cylinder. The rate of evaporation of the fuel from the puddle is assumed to be proportional to the puddle mass. A first order state model that describes this process is:

\[ \frac{dm_p}{dt} = -\frac{1}{\tau} m_p + X \dot{f}_i \]  

(10a)

where \(m_p\) is the mass of fuel in the puddle, \(\tau\) is the evaporation time constant of the fuel in the puddle, \(X\) is the fraction of the injected fuel that enters the puddle, and \(\dot{f}_i\) is the fuel rate out of the injectors. The rate at which fuel enters the cylinders \(\dot{f}_{cyl}\) is the sum of the rate at which fuel is directly injected and the rate at which fuel evaporates from the puddle:

\[ \dot{f}_{cyl} = \frac{1}{\tau} m_p + (1-X) \dot{f}_i. \]  

(10b)

Equations (10a) and (10b) provide a dynamic state model that relates the rate of fuel being supplied by the injectors to the rate of fuel entering the cylinders. The evaporation time constant in this model is a function of the intake manifold pressure, the engine speed, and the temperature of the intake port. The fraction of the injected fuel which enters the cylinders is also a function of the temperature of the intake valve and port. This model is, therefore, time-varying since the parameters \(\tau\) and \(X\) both vary during engine warm-up and during transient engine operating conditions. Nonlinearities also occur since fuel vaporization causes cooling of the intake port and, therefore, changes in the puddle dynamics.

4. The Air-Fuel Ratio Model
The state equations for the air flow and the fuel flow can be combined into a single state equation:

\[
\begin{bmatrix}
\frac{dp_i}{dt} \\
\frac{dm_{p}}{dt}
\end{bmatrix} = \begin{bmatrix}
-\frac{\eta V_c N}{2V_i} & 0 \\
0 & -\frac{1}{\tau}
\end{bmatrix} \begin{bmatrix}
p_i \\
m_p
\end{bmatrix} + \begin{bmatrix}
\frac{RT}{V_i} & 0 \\
0 & X
\end{bmatrix} \begin{bmatrix}
m_{\text{th}} \\
f_i
\end{bmatrix},
\]

(11)

The air-fuel ratio equals the mass rate of air entering the cylinders divided by the mass rate of fuel entering the cylinders:

\[
(A / F) = \frac{\sum m_{\text{cyl}}}{\dot{f}_{\text{cyl}}} = \frac{\frac{\eta V_d N}{2RT} p_i}{\frac{1}{\tau} m_p + (1 - X) \dot{f}_i}.
\]

(12a)

This output equation is not put in matrix form since it is highly nonlinear due to the division of the air flow rate by the fuel flow rate. An alternative output that is often used is the relative air-fuel ratio \( \lambda \). The relative air-fuel ratio is the actual air-fuel ratio divided by the stoichiometric air-fuel ratio:

\[
\lambda = \frac{(A / F)}{(A / F)_s} = \frac{\frac{\eta V_d N}{2RT(A / F)_s} p_i}{\frac{1}{\tau} m_p + (1 - X) \dot{f}_i}.
\]

(12b)

Either (12a) or (12b) can be used as an output of the model for air-fuel ratio.

Table 1 lists representative parameter values for the air-fuel ratio dynamics of the 1992 Ford 4.6 liter-2 valve Modular V8 engine. These parameters are only representative of the actual values which vary with engine and conditions. In addition, reasonable parameter variations are defined for cold start operation by providing a range for each parameter. These ranges are omitted for parameters with negligible variation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nomenclature</th>
<th>Nominal</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta^2 )</td>
<td>Volumetric Efficiency</td>
<td>0.80</td>
<td>0.67 - 0.94</td>
</tr>
<tr>
<td>( V_d )</td>
<td>Displacement Volume (liters)</td>
<td>4.6</td>
<td></td>
</tr>
<tr>
<td>( N )</td>
<td>Engine Speed (rev./sec.)</td>
<td>20.0</td>
<td>18.3 - 21.7</td>
</tr>
<tr>
<td>( V_i )</td>
<td>Intake Manifold Volume (liters)</td>
<td>7.36</td>
<td></td>
</tr>
<tr>
<td>( \tau^3 )</td>
<td>Evaporation Time Constant (sec.)</td>
<td>0.25</td>
<td>0.17 - 0.50</td>
</tr>
</tbody>
</table>

\(^2\) The volumetric efficiency varies as a function of intake manifold pressure and engine speed.

\(^3\) The evaporation time constant is a function of the manifold pressure and the temperature of the intake valve and the walls of the intake port.
<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>Gas Constant (J/kg·°K)</td>
<td>287</td>
<td>-</td>
</tr>
<tr>
<td>T</td>
<td>Intake Manifold Temp. (°K)</td>
<td>294</td>
<td>291 - 297</td>
</tr>
<tr>
<td>$X^4$</td>
<td>Fraction of Injected Fuel Entering the Puddle</td>
<td>0.6</td>
<td>0.3 - 0.9</td>
</tr>
</tbody>
</table>

Table 1. Parameter values and ranges for cold start.

\[4 \text{ The fraction of fuel entering the puddle is a function of the temperature of the intake valve and the walls in the intake port. This can be modeled as a time dependence during cold start.} \]
Appendix - Software Implementation Notes

A Modified Air Flow Model. The volumetric efficiency is a measure of the effectiveness of the engines intake process, and can be defined as the mass of air entering the cylinders during a single intake cycle divided by the perfect air charge:

\[ \eta = \frac{\Delta m_{cyl}}{p_i V_i \frac{RT}{N/2}} \]  \hspace{1cm} (A1)

where \( \Delta m_{cyl} \) is the mass of air entering the cylinders during a single intake cycle which is given:

\[ \Delta m_{cyl} = \frac{\sum_{cyl} \dot{m}_{cyl}}{N/2} \].  \hspace{1cm} (A2)

An alternative definition of the volumetric efficiency can be obtained by replacing the intake manifold pressure with standard atmospheric pressure in (A1):

\[ \eta_{std} = \frac{\Delta m_{cyl}}{p_{std} V_d \frac{RT}{N/2}} \].  \hspace{1cm} (A3)

The advantage of this substitution is that while \( \eta \) is a complicated nonlinear function of the intake manifold pressure, \( \eta_{std} \) can be approximated by an affine function of the intake manifold pressure:

\[ \eta_{std} = a \dot{p}_i + b \]  \hspace{1cm} (A4)

where \( a \) is a constant parameter and \( b \) is a parameter which depends on engine speed. Now, dividing (A1) by (A3) and solving for \( \eta \) in terms of \( \eta_{std} \) yields:

\[ \eta = \frac{p_{std}}{p_i} \eta_{std} \].  \hspace{1cm} (A5)

Substituting (A5) into the air flow state model:

\[ \frac{dp_i}{dt} = -\frac{p_{std} V_d N}{2V_i} \eta_{std} + \frac{RT}{V_i} \dot{m}_{th} \].  \hspace{1cm} (A6a)

\[ \sum_{cyl} \dot{m}_{cyl} = \frac{p_{std} V_d N}{2RT} \eta_{std} \].  \hspace{1cm} (A6b)

A new state model can then be obtained by substituting (A4) into (A6):
\[
\frac{dp_i}{dt} = \frac{p_{\text{std}} V_i N a_i}{2 V_i} p_i - \frac{p_{\text{std}} V_i N b_i}{2 V_i} + \frac{RT}{V_i} \dot{m}_{\text{th}}; \quad (A7a)
\]

\[
\sum_{\text{cyl}} \dot{m}_{\text{cyl}} = \eta_{\text{std}} \frac{V_i N a}{2RT} p_i \quad (A7b)
\]

This new state model includes an additional input $b$ which is a function of engine speed. The advantage of this state model is that it yields a simpler software implementation, and is, therefore, used in test cell 13. The values of $a$ and $b$ currently in use are:

\[
a = \frac{1}{p_{\text{std}}}; \quad (A8a)
\]

\[
b = \begin{cases} 
-0.1 & 800 \text{ RPM} \\
-0.06 & 1200 \text{ RPM} 
\end{cases} \quad (A8b)
\]

**Lost Fuel.** A number of studies have indicated that a portion of the fuel which enters the cylinders is lost, i.e., does not exit the tail pipe as either unburned fuel or combustion products. The most widely held explanation of this phenomena is that the fuel becomes entrained in the oil and winds up in the crankcase. This effect is most pronounced during cold start. The lost fuel effect can be incorporated in the state model for the fuel flow rate:

\[
\frac{dm_p}{dt} = -\frac{1}{\tau} m_p + X \dot{f}_i - \dot{f}_L; \quad (A9a)
\]

\[
\dot{f}_{\text{cyl}} = \frac{1}{\tau} m_p + (1 - X) \dot{f}_i \quad (A9b)
\]

where $\dot{f}_L$ is the rate of fuel loss which depends on the fuel injection rate:

\[
\dot{f}_L = \begin{cases} 
L X \dot{f}_i - \dot{f}_o & L X \dot{f}_i > \dot{f}_o \\
0 & L X \dot{f}_i \leq \dot{f}_o 
\end{cases} \quad (A10)
\]

The constant of proportionality $L$ and the offset $\dot{f}_o$ are parameters that are selected to match the data, and are approximately given:

\[
L = 0.5; \quad (A11a)
\]

\[
\dot{f}_o = 0.2 \text{ g / sec.} \quad (A11b)
\]