6.2 FIRST-ORDER LOWPASS FILTERS

Consider the circuit shown in Figure 6.7. We will see that this circuit tends to pass low-frequency components and reject high-frequency components. (In other words, for low frequencies, the output amplitude is nearly the same as the input. For high frequencies, the output amplitude is much less than the input.) In Chapter 4, we saw that a first-order differential equation describes this circuit. Because of these facts, the circuit is called a first-order lowpass filter.

To determine the transfer function, we apply a sinusoidal input signal having a phasor $V_{in}$, and then we analyze the behavior of the circuit as a function of the source frequency $f$.

The phasor current is the input voltage divided by the complex impedance of the circuit. This is given by

$$I = \frac{V_{in}}{R + 1/j2\pi f C} \quad (6.3)$$

The phasor for the output voltage is the product of the phasor current and the impedance of the capacitance, illustrated by

$$V_{out} = \frac{1}{j2\pi f C} I \quad (6.4)$$

Using Equation 6.3 to substitute for $I$, we have

$$V_{out} = \frac{1}{j2\pi f C} \times \frac{V_{in}}{R + 1/j2\pi f C} \quad (6.5)$$

Recall that the transfer function $H(f)$ is defined to be the ratio of the output phasor to the input phasor:

$$H(f) = \frac{V_{out}}{V_{in}} \quad (6.6)$$

Rearranging Equation 6.5, we have

$$H(f) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j2\pi f RC} \quad (6.7)$$

![Figure 6.7 A first-order lowpass filter.](image-url)
Next, we define the parameter:

\[ f_B = \frac{1}{2\pi RC} \]  
(6.8)

Then, the transfer function can be written as

\[ H(f) = \frac{1}{1 + j(f/f_B)} \]  
(6.9)

**Magnitude and Phase Plots of the Transfer Function**

As expected, the transfer function \( H(f) \) is a complex quantity having a magnitude and phase angle. Referring to the expression on the right-hand side of Equation 6.9, the magnitude of \( H(f) \) is the magnitude of the numerator (which is unity) over the magnitude of the denominator. Recall that the magnitude of a complex quantity is the square root of the sum of the real part squared and the imaginary part squared. Thus, the magnitude is given by

\[ |H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \]  
(6.10)

Referring to the expression on the right-hand side of Equation 6.9, the phase angle of the transfer function is the phase of the numerator (which is zero) minus the phase of the denominator. This is given by

\[ \angle H(f) = -\arctan\left(\frac{f}{f_B}\right) \]  
(6.11)

Plots of the magnitude and phase of the transfer function are shown in Figure 6.8. For low frequencies \( f \) approaching zero, the magnitude is approximately unity and the phase is nearly zero, which means that the amplitudes and phases of low-frequency components are affected very little by this filter. The low-frequency components are passed to the output almost unchanged in amplitude or phase.

![Figure 6.8 Magnitude and phase of the first-order lowpass transfer function versus frequency.](image-url)
On the other hand, for high frequencies \( f \gg f_R \), the magnitude of the transfer function approaches zero. Thus, the amplitude of the output is much smaller than the amplitude of the input for the high-frequency components. We say that the high-frequency components are rejected by the filter. Furthermore, at high frequencies, the phase of the transfer function approaches \(-90^\circ\). Thus as well as being reduced in amplitude, the high-frequency components are phase shifted.

Notice that for \( f = f_R \), the magnitude of the output is \( 1/\sqrt{2} \approx 0.707 \) times the magnitude of the input signal. When the amplitude of a voltage is multiplied by a factor of \( 1/\sqrt{2} \), the power that the voltage can deliver to a given resistance is multiplied by a factor of one-half (because power is proportional to voltage squared). Thus, \( f_R \) is called the **half-power frequency**.

### Applying the Transfer Function

As we saw in Section 6.1, if an input signal to a filter consists of several components of different frequencies, we can use the transfer function to compute the output for each component separately. Then, we can find the complete output by adding the separate components.

**Example 6.3 Calculation of \( RC \) Lowpass Output**

Suppose that an input signal given by

\[
v_{in}(t) = 5 \cos(20 \pi t) + 5 \cos(200 \pi t) + 5 \cos(2000 \pi t)
\]

is applied to the lowpass \( RC \) filter shown in Figure 6.9. Find an expression for the output signal.

**Solution** The filter has the form of the lowpass filter analyzed in this section. The half-power frequency is given by

\[
f_R = \frac{1}{2 \pi RC} = \frac{1}{2 \pi \times (1000/2 \pi) \times 10 \times 10^{-6}} = 100 \text{ Hz}
\]

The first component of the input signal is

\[
v_{in1}(t) = 5 \cos(20 \pi t)
\]

![Figure 6.9 Circuit of Example 6.3. The resistance has been picked so the break frequency turns out to be a convenient value.](image)
For this component, the phasor is $V_{\text{in}1} = 5 \angle 0^\circ$, and the angular frequency is $\omega = 20\pi$. Therefore, $f = \omega/2\pi = 10$. The transfer function of the circuit is given by

$$H(f) = \frac{1}{1 + j(f/f_b)}$$

Evaluating the transfer function for the frequency of the first component ($f = 10$), we have

$$H(10) = \frac{1}{1 + j(10/100)} = 0.9950 \angle -5.71^\circ$$

The output phasor for the $f = 10$ component is simply the input phasor times the transfer function. Thus, we obtain

$$V_{\text{out}1} = H(10) \times V_{\text{in}1}$$

$$= (0.9950 \angle -5.71^\circ) \times (5 \angle 0^\circ) = 4.975 \angle -5.71^\circ$$

Hence, the output for the first component of the input signal is

$$v_{\text{out}1}(t) = 4.975 \cos(20\pi t - 5.71^\circ)$$

Similarly, the second component of the input signal is

$$v_{\text{in}2}(t) = 5 \cos(200\pi t)$$

and we have

$$V_{\text{in}2} = 5 \angle 0^\circ$$

The frequency of the second component is $f = 100$:

$$H(100) = \frac{1}{1 + j(100/100)} = 0.7071 \angle -45^\circ$$

$$V_{\text{out}2} = H(100) \times V_{\text{in}2}$$

$$= (0.7071 \angle -45^\circ) \times (5 \angle 0^\circ) = 3.535 \angle -45^\circ$$

Therefore, the output for the second component of the input signal is

$$v_{\text{out}2}(t) = 3.535 \cos(200\pi t - 45^\circ)$$

Finally, for the third and last component, we have

$$v_{\text{in}3}(t) = 5 \cos(2000\pi t)$$

$$V_{\text{in}3} = 5 \angle 0^\circ$$

$$H(1000) = \frac{1}{1 + j(1000/100)} = 0.0995 \angle -84.29^\circ$$

$$V_{\text{out}3} = H(1000) \times V_{\text{in}3}$$

$$= (0.0995 \angle -84.29^\circ) \times (5 \angle 0^\circ) = 0.4975 \angle -84.29^\circ$$
Consequently, the output for the third component of the input signal is

$$v_{out3}(t) = 0.4975 \cos(2000\pi t - 84.29^\circ)$$

Now, we can write an expression for the output signal by adding the output components:

$$v_{out}(t) = 4.975 \cos(20\pi t - 5.71^\circ) + 3.535 \cos(200\pi t - 45^\circ) + 0.4975 \cos(2000\pi t - 84.29^\circ)$$

Notice that each component of the input signal $v_{in}(t)$ is treated differently by this filter. The $f = 10$ component is nearly unaffected in amplitude and phase. The $f = 100$ component is reduced in amplitude by a factor of 0.7071 and phase shifted by $-45^\circ$. The amplitude of the $f = 1000$ component is reduced by approximately an order of magnitude. Thus, the filter discriminates against the high-frequency components.

**Application of the First-Order Lowpass Filter**

A simple application of the first-order lowpass filter is the tone control on a simple AM radio. The tone control adjusts the resistance and, therefore, the break frequency of the filter. Suppose that we are listening to an interesting news item from a distant radio station with an AM radio and lightning storms are causing electrical noise. It turns out that the components of voice signals are concentrated in the low end of the audible-frequency range. On the other hand, the noise caused by lightning has roughly equal-amplitude components at all frequencies. In this situation, we could adjust the tone control to lower the break frequency. Then, the high-frequency noise components would be rejected, while most of the voice components would be passed. In this way, we can improve the ratio of desired signal power to noise power produced by the loudspeaker and make the news more intelligible.

**Using Phasors with Components of Different Frequencies**

Recall that phasors can be combined only for sinusoids with the same frequency. It is important to understand that we should not add the phasors for components with different frequencies. Thus, in the preceding example, we used phasors to find the output components as functions of time, which we then added.

**Exercise 6.4** Derive an expression for the transfer function $H(f) = V_{out}/V_{in}$ of the filter shown in Figure 6.10. Show that $H(f)$ takes the same form as Equation 6.9 if we define $f_B = R/2\pi L$.

**Exercise 6.5** Suppose that the input signal for the circuit shown in Figure 6.11 is given by

$$v_{in}(t) = 10 \cos(40\pi t) + 5 \cos(1000\pi t) + 5 \cos(2\pi 10^4 t)$$

Find an expression for the output signal $v_{out}(t)$.
Section 6.3  Decibels, the Cascade Connection, and Logarithmic Frequency Scales

Answer
\[ v_{\text{out}}(t) = 9.95 \cos(40\pi t - 5.71^\circ) + 1.86 \cos(1000\pi t - 68.2^\circ) + 0.100 \cos(2\pi 10^4t - 88.9^\circ) \]

6.3  DECIBELS, THE CASCADE CONNECTION, AND LOGARITHMIC FREQUENCY SCALES

In comparing the performance of various filters, it is helpful to express the magnitudes of the transfer functions in **decibels**. To convert a transfer-function magnitude to decibels, we multiply the common logarithm (base 10) of the transfer-function magnitude by 20:

\[ |H(f)|_{\text{dB}} = 20 \log |H(f)| \quad (6.12) \]

(A transfer function is a ratio of voltages and is converted to decibels as 20 times the logarithm of the ratio. On the other hand, ratios of powers are converted to decibels by taking 10 times the logarithm of the ratio.)

Table 6.2 shows the decibel equivalents for selected values of transfer-function magnitude. Notice that the decibel equivalents are positive for magnitudes greater than unity, whereas the decibel equivalents are negative for magnitudes less than unity.

In many applications, the ability of a filter to strongly reject signals in a given frequency band is of primary importance. For example, a common problem associated with audio signals is that a small amount of the ac power-line voltage can inadvertently be added to the signal. When applied to a loudspeaker, this 60-Hz component produces a disagreeable hum.
Table 6.2. Transfer-Function Magnitudes and Their Decibel Equivalents

| $|H(f)|$ | $|H(f)|_{\text{dB}}$ |
|-------|-----------------|
| 100   | 40              |
| 10    | 20              |
| 2     | 6               |
| $\sqrt{2}$ | 3           |
| 1     | 0               |
| $1/\sqrt{2}$ | -3        |
| $1/2$ | -6              |
| 0.1   | -20             |
| 0.01  | -40             |

Usually, we approach this problem by trying to eliminate the electrical path by which the power-line voltage is added to the desired audio signal. However, this is sometimes not possible. Then, we could try to design a filter that rejects the 60-Hz component and passes components at other frequencies. The magnitude of a filter transfer function to accomplish this is shown in Figure 6.12(a). A filter such as this, designed to eliminate components in a narrow range of frequencies, is called a notch filter.

It turns out that to reduce a loud hum (as loud as a heated conversation) to be barely audible, the transfer function must be $-80$ dB or less for the 60-Hz component, which corresponds to $|H(f)| = 10^{-4}$ or smaller. On the other hand, the transfer-function magnitude should be close to unity for the components to be passed by the filter. We refer to the range of frequencies to be passed as the passband.

When we plot $|H(f)|$ without converting to decibels, it is difficult to show both values clearly on the same plot. If we choose a scale that shows the passband magnitude, we cannot see whether the magnitude is sufficiently small at 60 Hz. This is the case for the plot shown in Figure 6.12(a). On the other hand, if we choose a linear scale that clearly shows the magnitude at 60 Hz, the magnitude would be way off scale at other frequencies of interest.

![Figure 6.12 Transfer-function magnitude of a notch filter used to reduce hum in audio signals.](image)
However, when the magnitude is converted to decibels, both parts of the magnitude are readily seen. For example, Figure 6.12(b) shows the decibel equivalent for the magnitude plot shown in Figure 6.12(a). On this plot, we can see that the passband magnitude is approximately unity (0 dB) and that at 60 Hz, the magnitude is sufficiently small (less than −80 dB).

Thus, one of the advantages of converting transfer-function magnitudes to decibels before plotting is that very small and very large magnitudes can be displayed clearly on a single plot. We will see that another advantage is that decibel plots for many filter circuits can be approximated by straight lines (provided that a logarithmic scale is used for frequency). Furthermore, to understand some of the jargon used by electrical engineers, we must be familiar with decibels.

**Cascaded Two-Port Networks**

When we connect the output terminals of one two-port circuit to the input terminals of another two-port circuit, we say that we have a **cascade** connection. This is illustrated in Figure 6.13. Notice that the output voltage of the first two-port network is the input voltage of the second two-port. The overall transfer function is

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}}$$

However, the output voltage of the cascade is the output of the second two port (i.e., $V_{\text{out}} = V_{\text{out}2}$). Furthermore, the input to the cascade is the input to the first two port (i.e., $V_{\text{in}} = V_{\text{in}1}$). Thus,

$$H(f) = \frac{V_{\text{out}2}}{V_{\text{in}1}}$$

Multiplying and dividing by $V_{\text{out}1}$, we have

$$H(f) = \frac{V_{\text{out}1}}{V_{\text{in}1}} \times \frac{V_{\text{out}2}}{V_{\text{out}1}}$$

Now, the output voltage of the first two port is the input to the second two port (i.e., $V_{\text{out1}} = V_{\text{in2}}$). Hence,

$$H(f) = \frac{V_{\text{out}1}}{V_{\text{in}1}} \times \frac{V_{\text{out}2}}{V_{\text{in}2}}$$

![Figure 6.13 Cascade connection of two two-port circuits.](image)
Finally, we can write

$$H(f) = H_1(f) \times H_2(f)$$  \hspace{1cm} (6.13)

Thus, the transfer function of the cascade connection is the product of the transfer functions of the individual two-port networks. This fact can be extended to three or more two ports connected in cascade.

A potential source of difficulty in applying Equation 6.13 is that the transfer function of a two port usually depends on what is attached to its output terminals. Thus, in applying Equation 6.13, we must find $H_1(f)$ with the second two port attached.

Taking the magnitudes of the terms on both sides of Equation 6.13 and expressing in decibels, we have

$$20 \log |H(f)| = 20 \log (|H_1(f)| \times |H_2(f)|)$$ \hspace{1cm} (6.14)

Using the fact that the logarithm of a product is equal to the sum of the logarithms of the terms in the product, we have

$$20 \log |H(f)| = 20 \log |H_1(f)| + 20 \log |H_2(f)|$$ \hspace{1cm} (6.15)

which can be written as

$$|H(f)|_{AB} = |H_1(f)|_{AB} + |H_2(f)|_{AB}$$ \hspace{1cm} (6.16)

Thus, in decibels, the individual transfer-function magnitudes are added to find the overall transfer-function magnitude for a cascade connection.

**Logarithmic Frequency Scales**

We often use a logarithmic scale for frequency when plotting transfer functions. On a logarithmic scale, the variable is multiplied by a given factor for equal increments of length along the axis. (On a linear scale, equal lengths on the scale correspond to adding a given amount to the variable.) For example, a logarithmic frequency scale is shown in Figure 6.14.

A **decade** is a range of frequencies for which the ratio of the highest frequency to the lowest is 10. The frequency range from 2 to 20 Hz is one decade. Similarly, the range from 50 to 5000 Hz is two decades. (50 to 500 Hz is one decade, and 500 to 5000 Hz is another decade.)

An **octave** is a two-to-one change in frequency. For example, the range 10 to 20 Hz is one octave. The range 2 to 16 kHz is three octaves.

![Logarithmic frequency scale](image-url)  

**Figure 6.14** Logarithmic frequency scale.
Suppose that we have two frequencies \( f_1 \) and \( f_2 \) for which \( f_2 > f_1 \). The number of decades between \( f_1 \) and \( f_2 \) is given by

\[
\text{number of decades} = \log \left( \frac{f_2}{f_1} \right) \quad (6.17)
\]

in which we assume that the logarithm is base 10. The number of octaves between the two frequencies is

\[
\text{number of octaves} = \log_2 \left( \frac{f_2}{f_1} \right) = \frac{\log(f_2/f_1)}{\log(2)} \quad (6.18)
\]

The advantage of a logarithmic frequency scale compared with a linear scale is that the variations in the magnitude or phase of a transfer function for a low range of frequency such as 10 to 20 Hz, as well as the variations in a high range such as 10 to 20 MHz, can be clearly shown on a single plot. With a linear scale, either the low range would be severely compressed or the high range would be off scale.

**Exercise 6.6** Suppose that \( |H(f)| = 50 \). Find the decibel equivalent.

**Answer** \( |H(f)|_{\text{dB}} = 34 \text{ dB} \).

**Exercise 6.7**

a. Suppose that \( |H(f)|_{\text{dB}} = 15 \text{ dB} \). Find \( |H(f)| \).

b. Repeat for \( |H(f)|_{\text{dB}} = 30 \text{ dB} \).

**Answer**

a. \( |H(f)| = 5.62 \)

b. \( |H(f)| = 31.6 \)

**Exercise 6.8**

a. What frequency is two octaves higher than 1000 Hz?

b. Three octaves lower?

c. Two decades higher?

d. One decade lower?

**Answer**

a. 4000 Hz is two octaves higher than 1000 Hz;
b. 125 Hz is three octaves lower than 1000 Hz;
c. 100 kHz is two decades higher than 1000 Hz;
d. 100 Hz is one decade lower than 1000 Hz.

**Exercise 6.9**

a. What frequency is halfway between 100 and 1000 Hz on a logarithmic frequency scale?

b. On a linear frequency scale?

**Answer**

a. 316.2 Hz is halfway between 100 and 1000 Hz on a logarithmic scale;
b. 550 Hz is halfway between 100 and 1000 Hz on a linear frequency scale.

**Exercise 6.10**

a. How many decades are between \( f_1 = 20 \text{ Hz} \) and \( f_2 = 15 \text{ kHz} \)?

(This is the approximate range of audible frequencies.)

b. How many octaves?

**Answer**

a. Number of decades = \( \log \left( \frac{15 \text{ kHz}}{20 \text{ Hz}} \right) = 2.87 \)

b. Number of octaves = \( \frac{\log(15000/20)}{\log(2)} = 9.55 \)
6.4 BODE PLOTS

A Bode plot uses a logarithmic scale for frequency to show the magnitude of a network function in decibels versus frequency. Because it can clearly illustrate very large and very small magnitudes for a wide range of frequencies on one plot, the Bode plot is particularly useful for displaying transfer functions. Furthermore, it turns out that Bode plots of network functions can often be closely approximated by straight-line segments, so they are relatively easy to draw. (Actually, we now use computers to plot functions, so this advantage is not as important as it once was.) Terminology related to these plots is frequently encountered in signal-processing literature. Finally, an understanding of Bode plots enables us to make estimates quickly when dealing with transfer functions.

To illustrate Bode plot concepts, we consider the first-order lowpass transfer function of Equation 6.9, repeated here for convenience:

\[ H(f) = \frac{1}{1 + j(f/f_b)} \]

The magnitude of this transfer function is given by Equation 6.10, which is

\[ |H(f)| = \frac{1}{\sqrt{1 + (f/f_b)^2}} \]

To convert the magnitude to decibels, we take 20 times the logarithm of the magnitude:

\[ |H(f)|_{\text{dB}} = 20 \log |H(f)| \]

Substituting the expression for the transfer-function magnitude, we get

\[ |H(f)|_{\text{dB}} = 20 \log \frac{1}{\sqrt{1 + (f/f_b)^2}} \]

Using the properties of the logarithm, we obtain

\[ |H(f)|_{\text{dB}} = 20 \log(1) - 20 \log \sqrt{1 + \left(\frac{f}{f_b}\right)^2} \]

Of course, the logarithm of unity is zero. Therefore,

\[ |H(f)|_{\text{dB}} = -20 \log \sqrt{1 + \left(\frac{f}{f_b}\right)^2} \]

Finally, since \( \log(\sqrt{x}) = \frac{1}{2} \log(x) \), we have

\[ |H(f)|_{\text{dB}} = -10 \log[1 + \left(\frac{f}{f_b}\right)^2] \]  \( \text{(6.19)} \)

The low-frequency asymptote is constant at 0 dB.

Notice that the value given by Equation 6.19 is approximately 0 dB for \( f << f_b \). Thus, for low frequencies, the transfer-function magnitude is approximated by the horizontal straight line shown in Figure 6.15, labeled as the low-frequency asymptote.
Table 6.3. Values of the Approximate Expression (Equation 6.20) for Selected Frequencies

| $f$    | $|H(f)|_{dB}$ |
|--------|---------------|
| $f_B$  | 0             |
| $2f_B$ | −6            |
| $10f_B$| −20           |
| $100f_B$| −40          |
| $1000f_B$| −60        |

On the other hand, for $f >> f_B$, Equation 6.19 is approximately

$$|H(f)|_{dB} \approx -20 \log \left( \frac{f}{f_B} \right) \quad (6.20)$$

Evaluating for various values of $f$, we obtain the results shown in Table 6.3. Plotting these values results in the straight line shown sloping downward on the right-hand side of Figure 6.15, labeled as the high-frequency asymptote. Notice that the two straight-line asymptotes intersect at the half-power frequency $f_B$. For this reason, $f_B$ is also known as the corner frequency or as the break frequency.

Also, notice that the slope of the high-frequency asymptote is $-20 \, \text{dB} \text{ per decade}$ of frequency. (This slope can also be stated as $-6 \, \text{dB} \text{ per octave}$.)

If we evaluate Equation 6.19 at $f = f_B$, we find that $|H(f_B)|_{dB} = -3 \, \text{dB}$

Thus, the asymptotes are in error by only 3 dB at the corner frequency. The actual curve for $|H(f)|_{dB}$ is also shown in Figure 6.15.

The high-frequency asymptote slopes downward at 20 dB/decade, starting from 0 dB at $f_B$.

Notice that the two straight-line asymptotes intersect at the half-power frequency $f_B$.

The asymptotes are in error by only 3 dB at the corner frequency $f_B$. 

\[ |H(f)|_{dB} \approx -20 \log \left( \frac{f}{f_B} \right) \]

Figure 6.15. Magnitude Bode plot for the first-order low-pass filter.
Figure 6.16 Phase Bode plot for the first-order low-pass filter.

Phase Plot

The phase of the first-order lowpass transfer function is given by Equation 6.11, which is repeated here for convenience:

$$\angle H(f) = -\arctan \left( \frac{f}{f_B} \right)$$

Evaluating, we find that the phase approaches zero at very low frequencies, equals $-45^\circ$ at the break frequency, and approaches $-90^\circ$ at high frequencies.

Figure 6.16 shows a plot of phase versus frequency. Notice that the curve can be approximated by the following straight-line segments:

1. A horizontal line at zero for $f < f_B/10$.
2. A sloping line from zero phase at $f_B/10$ to $-90^\circ$ at $10f_B$.
3. A horizontal line at $-90^\circ$ for $f > 10f_B$.

The actual phase curve departs from these straight-line approximations by less than $6^\circ$. Hence, working by hand, we could easily construct an approximate plot of phase.

Many circuit functions can be plotted by the methods we have demonstrated for the simple lowpass $RC$ circuit; however, we will not try to develop your skill at this to a high degree. Bode plots of amplitude and phase for $RLC$ circuits are easily produced by computer programs. We have shown the manual approach to analyzing and drawing the Bode plot for the $RC$ lowpass filter mainly to present the concepts and terminology.

Exercise 6.11 Sketch the approximate straight-line Bode magnitude and phase plots to scale for the circuit shown in Figure 6.17.

Answer See Figure 6.18.
Section 6.5 First-Order Highpass Filters

![Circuit Diagram]

Figure 6.17 Circuit for Exercise 6.11.

![20 log |Vout / Vin| and Phase Graphs]

Figure 6.18 Answers for Exercise 6.11.

6.5 FIRST-ORDER HIGHPASS FILTERS

The circuit shown in Figure 6.19 is called a first-order highpass filter. It can be analyzed in much the same manner as the lowpass circuit considered earlier in this chapter. The resulting transfer function is given by

\[ H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j(f/f_B)}{1 + j(f/f_B)} \]  

(6.21)

in which

\[ f_B = \frac{1}{2\pi RC} \]  

(6.22)

Exercise 6.12 Use circuit analysis to derive the transfer function for the circuit of Figure 6.19, and show that it can be put into the form of Equations 6.21 and 6.22.
Magnitude and Phase of the Transfer Function

The magnitude of the transfer function is given by

\[ |H(f)| = \frac{f/f_B}{\sqrt{1 + (f/f_B)^2}} \]  \hspace{1cm} (6.23)

This is plotted in Figure 6.20(a). Notice that the transfer-function magnitude goes to zero for dc \((f = 0)\). For high frequencies \((f >> f_B)\), the transfer-function magnitude approaches unity. Thus, this filter passes high-frequency components and tends to reject low-frequency components. That is why the circuit is called a highpass filter.

Highpass filters are useful whenever we want to retain high-frequency components and reject low-frequency components. For example, suppose that we want to record warbler songs in a noisy environment. It turns out that bird calls fall in the high-frequency portion of the audible range. The audible range of frequencies is from 20 Hz to 15 kHz (approximately), and the calls of warblers fall (mainly) in the range above 2 kHz. On the other hand, the noise may be concentrated at lower frequencies. For example, heavy trucks rumbling down a bumpy road would produce strong noise components lower in frequency than 2 kHz. To record singing warblers in the vicinity of such a noise source, a highpass filter would be helpful. We would select \(R\) and \(C\) to achieve a half-power frequency \(f_B\) of approximately 2 kHz. Then, the filter would pass the songs and reject some of the noise.

![Figure 6.20](image)

Figure 6.20 Magnitude and phase for the first-order highpass transfer function.
Recall that if the amplitude of a component is multiplied by a factor of \(1/\sqrt{2}\), the power that the component can deliver to a resistance is multiplied by a factor of \(1/2\). For \(f = f_B\), \(|H(f)| = 1/\sqrt{2} \approx 0.707\), so that, as in the case of the lowpass filter, \(f_B\) is called the half-power frequency. (Here again, several alternative names are corner frequency and break frequency.)

The phase of the highpass transfer function (Equation 6.21) is given by

\[
\phi H(f) = 90^\circ - \arctan \left( \frac{f}{f_B} \right)
\]  

(6.24)

A plot of the phase shift of the highpass filter is shown in Figure 6.20(b).

**Bode Plots for the First-Order Highpass Filter**

As we have seen, a convenient way to plot transfer functions is to use the Bode plot, in which the magnitude is converted to decibels and a logarithmic frequency scale is used. In decibels, the magnitude of the highpass transfer function is

\[
|H(f)|_{\text{dB}} = 20 \log \frac{f/f_B}{\sqrt{1 + (f/f_B)^2}}
\]

This can be written as

\[
|H(f)|_{\text{dB}} = 20 \log \left( \frac{f}{f_B} \right) - 10 \log \left[ 1 + \left( \frac{f}{f_B} \right)^2 \right]
\]

(6.25)

For \(f << f_B\), the second term on the right-hand side of Equation 6.25 is approximately zero. Thus, for \(f << f_B\), we have

\[
|H(f)|_{\text{dB}} \approx 20 \log \left( \frac{f}{f_B} \right) \quad \text{for } f << f_B
\]

(6.26)

Evaluating this for selected values of \(f\), we find the values given in Table 6.4. Plotting these values, we obtain the low-frequency asymptote shown on the left-hand side of Figure 6.21(a). Notice that the low-frequency asymptote slopes downward to the left at a rate of 20 dB per decade.

For \(f >> f_B\), the magnitude given by Equation 6.25 is approximately 0 dB. Hence,

\[
|H(f)|_{\text{dB}} \approx 0 \quad \text{for } f >> f_B
\]

(6.27)

This is plotted as the high-frequency asymptote in Figure 6.21(a). Notice that the high-frequency asymptote and the low-frequency asymptote meet at \(f = f_B\). (That is why \(f_B\) is sometimes called the break frequency.)

**Table 6.4. Values of the Approximate Expression Given in Equation 6.26 for Selected Frequencies**

| \(f\) | \(|H(f)|_{\text{dB}}\) |
|-------|-----------------|
| \(f_B\) | 0 |
| \(f_B/2\) | -6 |
| \(f_B/10\) | -20 |
| \(f_B/100\) | -40 |
The actual values of $|H(f)|_{\text{dB}}$ are also plotted in Figure 6.21(a). Notice that the actual value at $f = f_B$ is $|H(f_B)|_{\text{dB}} = -3$ dB. Thus, the actual curve is only 3 dB from the asymptotes at $f = f_B$. For other frequencies, the actual curve is closer to the asymptotes. The Bode phase plot is shown in Figure 6.21(b) along with straight-line approximations.

**Example 6.4  Determination of the Break Frequency for a Highpass Filter**

Suppose that we want a first-order highpass filter that has a transfer-function magnitude of $-30$ dB at $f = 60$ Hz. Find the break frequency for this filter.

**Solution**  Recall that the low-frequency asymptote slopes at a rate of 20 dB/decade. Thus, we must select $f_B$ to be

$$\frac{30 \text{ dB}}{20 \text{ dB/decade}} = 1.5 \text{ decades}$$

higher than 60 Hz. Employing Equation 6.17, we have

$$\log \left( \frac{f_B}{60} \right) = 1.5$$

This is equivalent to

$$\frac{f_B}{60} = 10^{1.5} = 31.6$$

which yields

$$f_B \approx 1900 \text{ Hz}$$

We often need a filter that greatly reduces the amplitude of a component at a given frequency, but has a negligible effect on components at nearby frequencies. The preceding example shows that to reduce the amplitude of a given component
by a large factor by using a first-order filter, we must place the break frequency far from the component to be rejected. Then, components at other frequencies are also affected. This is a problem that can only be solved by using more complex (higher order) filter circuits. We consider second-order filters later in the chapter.

**Exercise 6.13** Consider the circuit shown in Figure 6.22. Show that the transfer function of this filter is given by Equation 6.21 if the half-power frequency is defined to be \( f_B = R/2 \pi L \).

**Exercise 6.14** Suppose that we need a first-order \( RC \) highpass filter that reduces the amplitude of a component at a frequency of 1 kHz by 50 dB. The resistance is to be 1 kΩ. Find the half-power frequency and the capacitance.

**Answer** \( f_B = 316 \text{ kHz}, C = 503 \text{ pF} \).

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**Computer-Generated Bode Plots**

We have used manual methods to illustrate Bode plot concepts for simple filter circuits mainly to illustrate concepts. While manual methods can be extended to more complex circuits, it is often quicker and more accurate to use computers to produce Bode plots.

Suppose we have an \( RLC \) filter circuit, and we want to produce a Bode plot of its transfer function by using computer software. Before we start writing a program, we need to have some idea of what the results will be. This is helpful not only in selecting the frequency range to plot, but also in gaining confidence that we have not made programming errors.

Often, even a complex circuit can be readily analyzed at very high and at very low frequencies. At very low frequencies, the inductances behave as short circuits and the capacitances behave as open circuits, as we discussed in Section 4.2 of this book. After all, the impedance of an inductance is \( j \omega L \), which becomes very small in magnitude at low frequencies. Similarly, the impedance of a capacitance is \( 1/(j \omega C) \), which becomes very large at low frequencies. Thus, we can replace the inductances by shorts and the capacitances by opens and analyze the simplified circuit to determine the filter transfer function at low frequencies.

Similarly, at very high frequencies, the inductances become open circuits and the capacitances become shorts. By analyzing the circuit at very low and very