Topics for Today:

- Announcements
  - Software: online students - apply for ATP/ATPDraw license, verify licensing when you receive it by e-mail, and we will mail you the install CD.
  - Office: EERC 614. Phone: 906.487.2857
  - Recommended problems & all solutions: 13 solns now posted.
  - Homework Syst Op - due this Friday. Online students need more time?

Ongoing topics...
- Chapter 13 - Power system operation
  - Optimal Dispatch, Generator Scheduling
    - System including losses - use [B] loss coefficient matrix
    - Penalty Factors, related to partial of $P_{TL}$

Chapter 16 - Stability
- Dr. Mork’s lecture notes “System Stability” – See Week 13.
- Basic overview. Lead-in to EE6210.
EE 5200 - Assignment "DISPATCH"  
Due: Fri Dec 11th, 5pm
Fall Semester 2009 - Mork  
(Don’t wait to begin !)

Objectives: Constrained minimization via LaGrange multipliers, lossless optimal dispatch, correcting unacceptable bus voltages with shunt capacitor banks or shunt reactors.

1) Using the method of LaGrange multipliers, minimize the function \( f(x,y) = x^2 + y^2 \), subject to the constraint \( y = 2x + 8 \). Sketch out the function and the constraint, and explain in practical terms what you’ve done.

2) After running a load flow simulation, the voltage magnitude at a load bus is found to be 1.10 per unit. (Hint: since you will need to address 2 cases here and you may need to do this calculation again in the future, consider writing a Matlab program to do this calculation.)
   a) What type of shunt impedance should be added at this bus to pull the voltage down to 1.0 per unit?
   b) If the Thevenin impedance at this bus is \( Z = 4.0 + j6.0 \) per unit, calculate the value of this shunt impedance. If your calculation indicates more than one mathematical solution, indicate which one you would select and why.
   c) Repeat parts a) and b) for the case where a bus voltage of 0.85 per unit is to be raised to 1.03 per unit.
   d) Write a short explanation of how this technique can be used for planning studies. What bit of information is needed that might not be directly available to someone who is running the load flow studies?

3) Two power plants in a system are operating under economic dispatch constraints. For this simplistic case, line losses are neglected. Minimum output of each unit is 0.5 per unit; maximum is 5.0 per unit. Incremental operating costs of the plants in per unit on a 100 MVA base are:

\[
\frac{\partial C_1}{\partial P_{G1}} = 200P_{G1} + 900 \\
\frac{\partial C_2}{\partial P_{G2}} = 300P_{G2} + 800
\]

<table>
<thead>
<tr>
<th>System Lambda</th>
<th>( P_{G1} )</th>
<th>( P_{G2} )</th>
<th>( P_{G1} + P_{G2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 p.u.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.75 p.u.</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>5.0 p.u.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0 p.u.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) Determine how to optimally schedule the generation for the 4 cases given in the table above. Show your calculations. (The lambda value asked for is the “operative system lambda.”

b) Plot \( P_{G1} \) and \( P_{G2} \) for system load over the range 1.0 to 10.0 per unit. Clearly label the regions where the inequality constraints are “active” (i.e. minimum or maximum power output limits of an individual generator). Also label the region where only the equalities apply.

c) What exactly is lambda? Write a short explanation of what it means. Contrast between what the operative “system lambda” is and the lambdas of individual generators.
\[ \min \quad f(x, y) = x^2 + y^2 \]

Subject to:
\[ g(x, y) \Rightarrow y = 2x + 8 \]

Sohm
Before, without losses: \( P_G = \sum_{i=1}^{n} P_{ai} = P_L \)

With losses:

\[ P_G = P_L + P_{TL} \]

Constraint Equation:

\[ G = P_G - P_L - P_{TL} = 0 \]

Main problem is to find \( P_{TL} \).

From Kron, Kirchmeyer, George:

Losses in terms of \( P_{ai} \):

\[ P_{TL} = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} B_{ij} P_{aj} \]

\[ = [P_L]^T [B] [P_L] \]

\[ = [P_{a1}, P_{a2}, \ldots, P_{an}] [B] \left[ \begin{array}{c} P_{a1} \\ P_{a2} \\ \vdots \\ P_{an} \end{array} \right] \]

How to determine \( B \) constants?

(Many methods. Each utility to their own.)

1) Empirical Method: "Hill-Stevenson Method"

Run lots of load flows and examine sensitivity of all \( S_i \) to \( P_{ai} \)

Key: Find all \( \frac{\partial S_k}{\partial P_{ai}} \bigg|_{k=1,N} \)

Refer to method in book.
Once B coefficients are obtained:

Setting up the Lagrangian:

\[
\text{minimize: } \quad \mathbf{C} = \sum_{i=1}^{n} C_i \quad \text{(objective)}
\]

subject to: \quad \sum_{i=1}^{n} (P_G - P_L - P_{TL}) = 0 \quad \text{(constraint)}

\[
L = \sum_{i=1}^{n} C_i - \lambda \left( \sum_{i=1}^{n} (P_G - P_L - P_{TL}) \right)
\]

\[
\frac{\partial L}{\partial P_{Gi}} = C_i - \lambda \left(1 - \frac{\partial P_{TL}}{\partial P_{Gi}}\right) = 0
\]

For optimized condition, \quad \lambda = \lambda_i \quad \text{(all } \lambda_i \text{'s equal)}

From before, \quad \frac{\partial C_i}{\partial P_{Gi}} = 2 \lambda_i P_{Gi} + \beta_i

\[
\frac{\partial P_{TL}}{\partial P_{Gi}} = 2 \sum_{j=1}^{n} \beta_{ij} P_{Gj}
\]

Constraint Equation:

\[
\sum_{i=1}^{n} P_{Gi} - P_L - \sum_{i=1}^{n} \sum_{j=1}^{n} P_{Gi} \beta_{ij} P_{Gj} = 0
\]
Once $[B]$ is known:

$$PF_i = \frac{1}{1 - \sum_{j=1}^{n} B_{ij} P_{aq}} = \frac{\partial P_i}{\partial P_{aq}} \text{ Penalty Factor}$$

$$\alpha_i = \frac{\partial C_i}{\partial P_{aq}}$$

$$\frac{\partial C_i}{\partial P_{aq}} = 2\alpha_i P_{aq} + \beta_i$$

$$PF_i = 1 - \frac{\partial P_i}{\partial P_{aq}}$$

$$\left(1 - \frac{\partial P_i}{\partial P_{aq}}\right)\alpha_i = \frac{\partial C_i}{\partial P_{aq}}$$
For optimum, $\lambda = \lambda_i$ (all $\lambda$s the same)

$$\lambda_i = \left( \frac{\partial \psi}{\partial P_{Gi}} \right) (PF_i) = \lambda_{i_{(sys)}}$$

Note: Error on p. 320 of book. $\lambda_i = \frac{2 \lambda_i P_Gi + \beta_i}{P_{F_i}}$

$$\lambda_i = \frac{2 \lambda_i P_Gi + \beta_i}{1 - 2 \sum_{j=1}^{n} \beta_{ij} P_{Gj}}$$

$$[\beta] = \begin{bmatrix} 0.001694 & -0.004940 \\ -0.004940 & 0.014406 \end{bmatrix}$$

$$\lambda_1 = \frac{2 \lambda_1 P_G1 + \beta_1}{1 - 2 (\beta_{11} P_G1 + \beta_{12} P_G2)}$$

$$\lambda_2 = \frac{2 \lambda_2 P_G2 + \beta_2}{1 - 2 (\beta_{21} P_G1 + \beta_{22} P_G2)}$$

$\lambda_1 = 100$ (approx.)

$\beta_1 = 200$

$\lambda_2 = 40$

$\beta_2 = 260$
\[ J_1 = \frac{200 \cdot P_{g1} + 200}{1 - 2 \left[ (0.001694) P_{g1} + (0.0044) P_{g2} \right]} \]

\[ J_2 = \frac{80 \cdot P_{g2} + 260}{1 - 2 \left[ (-0.00494) P_{g1} + (0.014406) P_{g2} \right]} \]

\[ P_{TL} = B_{11} P_{g1}^2 + 2 B_{12} P_{g1} P_{g2} + B_{22} P_{g2}^2 \]

\[ P_{g1} + P_{g2} - P_{TL} = 2.82 \]

Solve trial & error; (or could set up N-R)

(Could use starting values from case where line losses are neglected.)
i.e. - starting values of \( P_{g1} \) & \( P_{g2} \)

Start: \( P_{g1} = 1.02 \) p.m. \( P_{g2} = 1.80 \) p.m.

then calculate \( J_1, J_2, P_{TL} \).

If \( J_2 > J_1 \) then decrease \( P_{g2} \), increase \( P_{g1} \)

Calculate \( P_{TL} \) & \( P_{g1} + P_{g2} - P_{TL} \).

If \( P_{TL} < 2.82 \), increase both \( P_{g1} \) & \( P_{g2} \).

See table on page 320.

Hand calculation recommended to give students a feel for problem & interrelationships.
8.2 Economic Dispatch Considering Losses

In section 8.1, the economic dispatch problem was solved neglecting transmission losses. Experience has shown that in some cases, this approximation produces results that are in serious error. We recall the power balance equation

\[ P_G = P_L + P_{TL} \]  

(8.7a)

The revised equation of constraint, considering losses, is then

\[ G = P_G - P_L - P_{TL} = 0 \]  

(8.14)

Recall that the problem's variables are the \( P_{Gi} \)'s. Therefore, it is necessary to formulate the losses using the \( P_{Gi} \)'s as variables.

Let us consider the losses of the components of the transmission system, specifically, transformers and lines. Transformers have two types of losses: copper and iron. The iron, or magnetic, losses vary with core flux density, which in turn varies with voltage. Since the transformer voltage varies indirectly, and very little, with load, the variation of core loss with load is quite small and not important. However, the copper loss varies as \( I^2R \); the load current varies directly with power loading; and the series resistance loss varies with the square of this current. Thus, the transformer copper loss should be included in the transmission losses. Likewise, the \( I^2R \) losses in the series element of the transmission line constitute transmission losses. The challenge is to functionally relate these \( I^2R \) losses to the \( P_{Gi} \) variables.

Many investigators, including Kron, Kirchmeyer, and George have worked on this problem and proposed a loss equation that formulates the loss as a quadratic function of the \( P_{Gi} \)'s; \( ^* \)

\[ P_{TL} = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{Gi}B_{ij}P_{Gj} \]  

(8.15a)

\[ = \bar{P}_G[B]P_G \]  

(8.15b)

where

\[ \bar{P}_G = \begin{bmatrix} P_{G1} \\ P_{G2} \\ \vdots \\ P_{Gn} \end{bmatrix} \]  

vector of generated powers.

\[ [B] = (n \times n) \text{ array with general entry } B_{ij} \]

The \([B]\) array is symmetrical, such that \( B_{ij} = B_{ji} \). Computing the \( B_{ij} \) values, called the \( B \) constants, can be implemented by several techniques. One approach, called the Hill–Stevenson method, calculates \( B \) constants using partial derivatives,

\[ P_{TL} = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{Gi}B_{ij}P_{Gj} + \sum_{i=1}^{n} B_{i}P_{Gi} + B_0 \]

* A more general version is

\[ P_{TL} = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{Gi}B_{ij}P_{Gj} + \sum_{i=1}^{n} B_{i}P_{Gi} + B_0 \]

some of which are evaluated numerically using data obtained from a series of load flow runs. A description of the method follows. Observe that

\[ \frac{\partial P_{TL}}{\partial P_{Gi}} = \frac{\partial}{\partial P_{Gi}} \left[ \sum_{j=1}^{n} \sum_{j=1}^{n} P_{Gi}B_{ij}P_{Gj} \right] \]  

(8.16a)

\[ = 2 \sum_{j=1}^{n} B_{ij}P_{Gj} \]  

(8.16b)

Furthermore,

\[ \frac{\partial^2 P_{TL}}{\partial P_{Gi} \partial P_{Gj}} = 2B_{ij} = 2B_{ji} \]  

(8.17a)

But the subscript notation is arbitrary, so that

\[ B_{ij} = \frac{1}{2} \left[ \frac{\partial^2 P_{TL}}{\partial P_{Gi} \partial P_{Gj}} \right] \]  

(8.17b)

Consider the general power system as we viewed it for power flow in Figure 7.2. The total power injected into the transmission network is

\[ P_{TL} = \text{Re} \left[ \sum_{i=1}^{b} \bar{P}_i \bar{V}_i \right] \]  

(8.18a)

If the transmission network were lossless, \( P_{TL} \) would be zero (power in = power out); in general, this summation would equal the total transmission system loss \( P_{TL} \).

\[ P_{TL} = \text{Re} \left[ \sum_{i=1}^{b} \bar{P}_i \left( \sum_{j=1}^{b} \bar{V}_j \right) \right] \]  

(8.18b)

\[ = \sum_{i=1}^{b} \sum_{j=1}^{b} \bar{V}_i \bar{V}_j \cos(\delta_i - \delta_j - \gamma_{ij}) \]  

(8.18c)

Now, we compute from equation (8.18c)\( ^* \):

\[ \frac{\partial P_{TL}}{\partial \delta_k} = 2 \sum_{i=1}^{b} \bar{V}_i \bar{V}_k \sin(\delta_i - \delta_k) \]  

(8.19)

Now, by the chain rule, consider that

\[ \frac{\partial P_{TL}}{\partial P_{Gi}} = \sum_{k=1}^{b} \frac{\partial P_{TL}}{\partial \delta_k} \frac{\partial \delta_k}{\partial P_{Gi}} \]  

(8.20)

\( ^* \)Here, \( b \) is the total number of buses, and \( n \) generators. In Chapter 7, \( n \) was the total number of buses.

\( ^* \)The simplification is complicated and relegated to an exercise for the student. See problem 8.7.
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The troublesome terms are the $\partial \delta_k / \partial P_{g_i}$ factors. The Hill–Stevenson method provides a clever way of evaluating these partial derivatives numerically. A base load flow case is established that hopefully approximates the ultimate allocation of generation. One approach is to solve the lossless economic dispatch problem as a first approximation. Then increase all bus loads proportionally by a small amount (say, 10%), and allow generator 1 alone to pick up the increased load. The load flow solves for the phase ($\delta_k$) at all $b$ buses, from which the change in phase ($\Delta \delta_k$) can be calculated. Therefore,

$$\frac{\partial \delta_k}{\partial P_{g_i}} \approx \frac{\Delta \delta_k}{\Delta P_{g_i}} = A_{kj}$$

This partial derivative is reasonably constant over a rather wide range of generation and load values. We repeat the process $n$ times, allowing each generator to act alone to pick up the increase in load, such that

$$\frac{\partial \delta_k}{\partial P_{g_i}} \approx \frac{\Delta \delta_k}{\Delta P_{g_i}} = A_{kj} \quad k = 1, 2, \ldots, b \quad j = 1, 2, \ldots, n$$

(8.21)

The partial derivatives so approximated are represented as the $A$ constants. Let us now compute a second partial derivative

$$\frac{\partial}{\partial P_{g_i}} \left( \frac{\partial P_{TL}}{\partial P_{g_j}} \right) = \frac{\partial^2 P_{TL}}{\partial P_{g_i} \partial P_{g_j}} = \frac{\partial}{\partial P_{g_i}} \left[ \sum_{k=1}^{b} \frac{\partial P_{TL} (\delta_{k})}{\partial P_{g_j}} \right]$$

(8.22a)

$$= \sum_{m=1}^{b} \frac{\partial}{\partial \delta_m} \left( \sum_{k=1}^{b} \frac{\partial P_{TL} (\delta_{k})}{\partial \delta_k} \right) \left( \frac{\partial \delta_m}{\partial P_{g_j}} \right)$$

$$= \sum_{m=1}^{b} \sum_{k=1}^{b} \frac{\partial^2 P_{TL}}{\partial \delta_m \partial \delta_k} \frac{\partial \delta_m}{\partial P_{g_j}} \frac{\partial \delta_k}{\partial P_{g_i}}$$

or

$$\frac{\partial^2 P_{TL}}{\partial P_{g_i} \partial P_{g_j}} = \sum_{m=1}^{b} \sum_{k=1}^{b} \frac{\partial^2 P_{TL}}{\partial \delta_m \partial \delta_k} A_{mj} A_{ki}$$

(8.22b)

Recall that

$$\frac{\partial P_{TL}}{\partial \delta_k} = 2 \sum_{i=1}^{b} V_i V_i g_{ik} \sin(\delta_i - \delta_k)$$

(8.19)

We wish to compute

$$\frac{\partial}{\partial \delta_m} \left( \frac{\partial P_{TL}}{\partial \delta_k} \right) = \frac{\partial^2 P_{TL}}{\partial \delta_m \partial \delta_k}$$

(8.23a)

where $\delta_m$ is one specific $\delta_i$. We determine that

$$\frac{\partial^2 P_{TL}}{\partial \delta_m \partial \delta_k} = 2 V_m V_k g_{mk} \cos(\delta_m - \delta_k) \quad m \neq k$$

(8.23b)

$$= -2 \sum_{i=1}^{b} V_i V_m g_{im} \cos(\delta_i - \delta_m) \quad m = k$$

(8.23c)

To relate this work to the $B$ constants, recall that

$$B_{ij} = \frac{1}{2} \left( \frac{\partial^2 P_{TL}}{\partial P_{g_i} \partial P_{g_j}} \right)$$

(8.24)

Finally,

$$B_{ij} = \frac{1}{2} \left( \sum_{m=1}^{b} \sum_{k=1}^{b} \frac{\partial^2 P_{TL}}{\partial \delta_m \partial \delta_k} A_{mi} A_{kj} \right)$$

(8.25)

We now return to our main concern, that is, solving the economic dispatch problem considering losses.

$$C = \sum_{i=1}^{n} C_i \quad \text{(objective function)}$$

(8.6)

$$P_g - P_L - P_{TL} = 0 \quad \text{(equation of constraint)}$$

(8.14)

$$\mathcal{L} = \sum_{i=1}^{n} C_i - \lambda [P_g - P_L - P_{TL}] \quad \text{(the Lagrangian)}$$

(8.26)

The general partial derivative with respect to $P_{g_i}$ is

$$\frac{\partial \mathcal{L}}{\partial P_{g_i}} = \frac{\partial C_i}{\partial P_{g_i}} - \lambda \left( 1 - \frac{\partial P_{TL}}{\partial P_{g_i}} \right) = 0$$

(8.27)

or, we redefine $\lambda$, such that

$$\lambda_i = \frac{\partial C_i / \partial P_{g_i}}{\left( \partial P_{TL} / \partial P_{g_i} \right)} = \lambda_i$$

(8.28a)

where

$$\frac{\partial C_i}{\partial P_{g_i}} = 2 \alpha_i P_{g_i} + \beta_i$$

(8.28b)

and

$$\frac{\partial P_{TL}}{\partial P_{g_i}} = 2 \sum_{j=1}^{n} B_{ij} P_{g_j}$$

(8.28c)
and the equation of constraint requires that
\[
\left( \sum_{i=1}^{n} P_{G_i} \right) - P_L - \left( \sum_{i=1}^{n} \sum_{j=1}^{n} P_{G_i} B_{ij} P_{G_j} \right) = 0 \tag{8.29}
\]
Thus, the condition for economically optimum operation requires the weighted incremental cost functions for all units to be equal (to each other and \( \lambda \))
\[
\lambda_i = \lambda \tag{8.30}
\]
The weighting factor is sometimes called the penalty factor of generator \( i(PF_i) \)
\[
\lambda_i \left( PF_i \right) = \frac{1 - \left( \frac{\partial P_L}{\partial P_{G_i}} \right)}{1 - 2 \sum_{j=1}^{n} B_{ij} P_{G_j}} \tag{8.31}
\]
Example 8.3 is useful for illustrative purposes.

**Example 8.3**

A single-line diagram for the system in example 8.2 is shown in Figure 8.4. A base case load flow study on the system provides the following results:
- \( V_1 = 1/0^\circ \) \( P_{G_1} = 1.0313 \) \( P_{L1} = 2.1000 \)
- \( V_2 = 1/6.616^\circ \) \( P_{G_2} = 1.8200 \) \( P_{L2} = 0.7200 \)
- \( P_{TL} = 0.0313 \)

\[
\begin{align*}
&\text{All values in per-unit} \\
&\text{Figure 8.4. System for example 8.3.}
\end{align*}
\]
(a) Calculate the \( A \) constants. The load at each bus was increased 10%. If unit 1 picks up the load,
\[
\Delta \delta_1 = 0 \text{ (using bus 1 as phase reference)}
\]
\[
\Delta \delta_2 = 6.187 - 6.616 = -0.429^\circ \ (\text{0.007487 rad})
\]
\[
\Delta P_{G_1} = 1.3094 - 1.0313 = 0.2781
\]
\[
A_{11} = 0 \quad A_{21} = \frac{-0.007487}{0.278100} = -0.026924
\]

If unit 2 picks up the load,
\[
\Delta \delta_1 = -7.947 + 6.616 = -1.331^\circ \ (\text{0.02323 rad})
\]
\[
\Delta \delta_2 = 0 \text{ (using bus 2 as phase reference)}
\]
\[
\Delta P_{G_2} = 2.1159 - 1.8200 = 0.2959
\]
\[
A_{12} = \frac{-0.02323}{0.2959} = -0.078507 \quad A_{22} = 0
\]

(b) Calculate the \( B \) constants
\[
[Y] = \begin{bmatrix}
\end{bmatrix}
\]
\[
\begin{align*}
&g_{11} = g_{22} = 2.353 \\
&g_{12} = g_{21} = -2.353
\end{align*}
\]
\[
m = k
\]
\[
\frac{1}{2}(\partial^2 P_{TL}/\partial \delta_m \partial \delta_k) = - \sum_{l=1}^{2} \sum_{m=1}^{2} V_l V_m g_{lm} \cos(\delta_l - \delta_m)
\]
\[
= -(1)(1)(g_{12}) \cos(0 - 6.616^\circ)
\]
\[
= +2.337
\]
\[
m \neq k
\]
\[
\frac{1}{2}(\partial^2 P_{TL}/\partial \delta_m \partial \delta_k) = V_m V_k g_{mk} \cos(\delta_m - \delta_k)
\]
\[
= (1)(-2.337) = -2.337
\]

Finally,
\[
B_{ij} = \frac{1}{2} \sum_{m=1}^{2} \sum_{k=1}^{2} (\partial^2 P_{TL}/\partial \delta_m \partial \delta_k) A_{mi} A_{kj}
\]
\[
= 2.337(A_{11}A_{11} - A_{11}A_{22} - A_{22}A_{11} + A_{22}A_{22})
\]
\[
B_{11} = 2.337[(-0.026924)^2] = 0.001694
\]
\[
B_{12} = 2.337[(-0.026924)(-0.078507)] = -0.00493
\]
\[
B_{22} = 2.337[(-0.078507)^2] = 0.014406
\]

Check \( P_{TL} \)
\[
P_{TL} = B_{11}P_{G_1}^2 + 2B_{12}P_{G_1}P_{G_2} + B_{22}P_{G_2}^2
\]
\[
= (0.001694)(1.0313)^2 - 2(0.004940)(1.0313)(1.82) + (0.014406)(1.82)^2
\]
\[
= 0.0310 \text{ (compared with 0.0313; the small error is due to the fact that the small angle changes were in the order of the load flow convergence criteria.)}
\]
8.2 Economic Dispatch Considering Losses

(c) Solve for the economically optimum division of load considering losses. The penalty factors are

\[ PF_1 = \frac{1}{1 - (\partial P_L / \partial P_{G_1})} = \frac{1}{1 - 0.003388P_{G_1} + 0.009881P_{G_2}} \]

\[ PF_2 = \frac{1}{1 + 0.009881P_{G_1} - 0.02881P_{G_2}} \]

\[ \lambda_1 = \frac{PF_1}{P_{G_1} + 1} \]

\[ \lambda_2 = \frac{200P_{G_1} + 260}{1 + 0.009881P_{G_1} - 0.02881P_{G_2}} \]

The problem is solved by trial and error using a programmable calculator. The results of three trials are:

<table>
<thead>
<tr>
<th>( P_{G_1} )</th>
<th>( P_{G_1} )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( P_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0313</td>
<td>1.8200</td>
<td>400.4</td>
<td>423.5</td>
<td>2.820</td>
</tr>
<tr>
<td>1.1100</td>
<td>1.7400</td>
<td>416.4</td>
<td>415.5</td>
<td>2.823</td>
</tr>
<tr>
<td>1.1060</td>
<td>1.7410</td>
<td>415.6</td>
<td>415.6</td>
<td>2.820</td>
</tr>
</tbody>
</table>

The general problem is too complex to solve by hand. A flow chart for computer implementation is shown in Figure 8.5. The problem may also be formulated and solved by Newton–Raphson methods.

The foregoing discussion applies basically to an all-thermal system; that is, one where all generating units can be assigned an operation cost function \( C_i \). If hydro units are available, there is no fuel cost associated with the unit output and hence a negligible \( C_i \) function. However, this does not mean that there are no restrictions on hydro output whatsoever. The typical restriction involves the volume of water that may be removed from a reservoir in a specific time period. Consider

\[ P_H = \eta H g \rho \frac{dv}{dt} \]  \hspace{1cm} (8.32a)

where

- \( P_H \) = hydraulic turbine power in W.
- \( \eta \) = hydraulic efficiency of turbine and penstock.
- \( g \) = acceleration of gravity = 9.8 m/s\(^2\).
- \( H \) = difference in elevation of reservoir surface and turbine (head) in m.
- \( \rho \) = mass density of water = 1000 kg/m\(^3\).
- \( \frac{dv}{dt} \) = volume of water flow through turbine in m\(^3\)/s.

For a constant-head, constant-efficiency situation,

\[ P_H = K \frac{dv}{dt} \]  \hspace{1cm} (8.32b)

\[ K = H g \rho \]  \hspace{1cm} (8.32c)

so that

\[ \int \frac{dv}{K} = \frac{1}{K} \int P_H \, dt \]  \hspace{1cm} (8.33a)