Topics for Today:

- Announcements
  - Term Project final report due Fri Dec 12\textsuperscript{th} (extended to 9am ET Monday)
  - Final Project presentations - by Wed Dec 17\textsuperscript{th} 12:45-2:45pm (come early).
  - All teams prepare .ppt (with audiostream).
  - Office: EERC 614. Phone: 906.487.2857

- Project Presentations (on-campus teams)
  - Emphasize your project (Journal paper in Appdx of report)
  - 6 presentations in 2 hrs - \textasciitilde15-20 mins each including Q&A.
  - Provide .ppt handouts for audience (15 copies).
  - Handouts - landscape, 4 slides/page, double-sided, stapled

- Today:
  - Closing comments on stability
  - Intro to overall system operation (EE5230)
  - Quick look at State Estimation (Ch.15).
  - Intro to Smart Grid Technologies
  - Semester Wrapup
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*Stands for On-Campus On-line Student
**Stands for Off-Campus On-line Student
- Quick review of EE4221 + EE4222
- Delve more deeply into
  - Applied math
  - Concepts
  - Software: [Aspen, Matlab, ATP]
- Global perspective of power system.
- Professional skill development.

SMART GRID
SMART GRID
1) Advanced sensor "system" (fully)
2) Embedded processors
3) Comm & Control - GPS time-stamped
   - Wide Area

5223 - Section DB
5224 - Lab:
- State Estimation (EE5230)
  ↓
Synchro-Phasors
  ↓
PMU
  ↓
WAM / WAC / WAMPAC

SMART GRID
"Separate System"

→ Islands.
WAM - Wide Area Meas.
WAC - Wide Area Control

PMU - Phasor Measurement Unit

- Synchrophasors
  - High-speed updates

IEC 61850 - Inter-operability
  - High-speed intranet (10 Gbps)

IEEE 1588 - Data Synchronization
  - GPS time-tag

SCADA 1-2 s updates
- EE 6210 - Stability
- EE 5220 - Transients
- EE 5223 - Relaying

State Est. - Ch. 16

WAM - time-tag (GPS)
TYPICAL BASIC POWER SYSTEM

Generator that models external system

1. Generators
2. Transformers
3. Transmission Lines
4. Circuit Breakers
5. Loads

MAJOR COMPONENTS

LOCALLY CONTROLLABLE QTY

Voltage
Power (Voltage Angle)
Power Transfer (REAL & IM)
IN or OUT
OPEN or CLOSED

Limits only - loads fluctuate periodically.

System is modelled with R's, L's, C's and ac voltage or current sources. System is decidedly nonlinear and analysis done with complex variables either in polar or rectangular form.
SYSTEM ANALYSIS

I. LOADFLOW CALCULATIONS: Use Newton-Raphson method adapted to complex variables. Solution converges in 3-7 iterations regardless of system size. Solution gives voltage magnitude and angle at each bus, complex power flow in each bus connection, and generator outputs.

II. SYSTEM STABILITY: Numerical integration of 2nd order differential equations. Use loadflow to determine initial conditions of system. Solve for internal torque angle of each generator vs. time following a system disturbance. Can build up knowledge base for different system configurations (contingencies).

NEITHER OF ABOVE CAN BE PERFORMED ON REAL TIME BASIS, BUT CAN BE USED TO BUILD UP A KNOWLEDGE BASE FOR SYSTEM A.I. IMPLEMENTATIONS.
STATE ESTIMATION: Measurement devices (transducers) may fail or suffer a drift in calibration. Communication links may fail.

Fixup: 1) Substitute "pseudo-measurements" for unknown data based on available related measurements.
2) Perform WEIGHTED LEAST SQUARES ESTIMATION of all state variables.

\[
\text{Minimize } \sum_{i=1}^{m} \frac{(Z_i - f_i(x))^2}{C_i^2}
\]

- \( Z_i \): measured value
- \( f_i(x) \): estimated value
- \( C_i \): \( \gamma_{3\sigma} \) device tolerance (STD) \( (\pm 3\sigma \rightarrow 99\% \text{ of range}) \)

Solve using Newton–Raphson

3) IF \( |Z_i - f_i(x)| > 3\sigma \rightarrow \text{BAD MEASUREMENT} \)
4) Pass on estimated values to:

\[ \{ \text{CONTINGENCY ANALYSIS PROGRAM, CORRECTIVE ACTION PROGRAM} \} \]
MAJOR OPERATING PROBLEMS

1) Alarm Processing
2) Bad Data Elimination
3) Missing Data
4) Fault Diagnosis
5) Contingency Selection & Security Analysis
6) Normal Control
7) Preventative control
8) Emergency Control
9) Restorative Control
10) Unit Commitment
11) Maintenance Scheduling
12) Fuel Scheduling & Contracts
13) Power exchanges and Prices
14) Operator Training
15) Demand Management
16) Automated Manuals
2 POWER SYSTEM STATE ESTIMATION

introduced in Chapter 11, the problem of monitoring the power flows and
tages on a transmission system is very important in maintaining system
urity. By simply checking each measured value against its limit, the power
tem operators can tell where problems exist in the transmission system—and,
s hoped, they can take corrective actions to relieve overloaded lines or
of-limit voltages.

Many problems are encountered in monitoring a transmission system. These
blems come primarily from the nature of the measurement transducers and
communications problems in transmitting the measured values back to
operations control center.

Transducers from power system measurements, like any measurement device,
be subject to errors. If the errors are small, they may go undetected and
cause misinterpretation by those reading the measured values. In addition,
sducers may have gross measurement errors that render their output
less. An example of such a gross error might involve having the transducer
ected up backward; thus, giving the negative of the value being measured.
ually, the telemetry equipment often experiences periods when communica-
channels are completely out; thus, depriving the system operator of any
formation about some part of the power system network.

It is for these reasons that power state estimation techniques have been
developed. A state estimator, as we will see shortly, can “smooth out” small
dom errors in meter readings, detect and identify gross measurement errors,
“fill in” meter readings that have failed due to communications failures.
To begin, we will use a simple DC load flow example to illustrate the
iples of state estimation. Suppose the three-bus DC load flow of Example
were operating with the load and generation shown in Figure 12.1. The only
formation we have about this system is provided by three MW power flow
ers located as shown in Figure 12.2.

Only two of these meter readings are required to calculate the bus phase
gle and all load and generation values fully. Suppose we use $M_{13}$ and $M_{32}$
urther suppose that $M_{13}$ and $M_{32}$ give us perfect readings of the flows on
respective transmission lines.

\[
M_{13} = 5 \text{ MW} = 0.05 \text{ pu}
\]
\[
M_{32} = 40 \text{ MW} = 0.40 \text{ pu}
\]

In, the flows on lines 1-3 and 3-2 can be set equal to these meter readings.

\[
f_{13} = \frac{1}{x_{13}} (\theta_1 - \theta_3) = M_{13} = 0.05 \text{ pu}
\]
\[
f_{32} = \frac{1}{x_{32}} (\theta_3 - \theta_2) = M_{32} = 0.40 \text{ pu}
\]

Since we know that $\theta_3 = 0 \text{ rad}$, we can solve the $f_{13}$ equation for $\theta_1$, and the
$f_{32}$ equation for $\theta_2$, resulting in

\[
\theta_1 = 0.02 \text{ rad}
\]
\[
\theta_2 = -0.10 \text{ rad}
\]
We will now investigate the case where all three meter readings have slight errors. Suppose the readings obtained are

\[ M_{12} = 62 \text{ MW} = 0.62 \text{ pu} \]
\[ M_{13} = 6 \text{ MW} = 0.06 \text{ pu} \]
\[ M_{32} = 37 \text{ MW} = 0.37 \text{ pu} \]

If we use only the \( M_{13} \) and \( M_{32} \) readings, as before, we will calculate the phase angles as follows:

\[ \theta_1 = 0.024 \text{ rad} \]
\[ \theta_2 = -0.0925 \text{ rad} \]
\[ \theta_3 = 0 \text{ rad (still assumed to equal zero)} \]

This results in the system flows as shown in Figure 12.3. Note that the predicted flows match at \( M_{13} \) and \( M_{32} \), but the flow on line 1-2 does not match the reading of 62 MW from \( M_{12} \). If we were to ignore the reading on \( M_{13} \) and use \( M_{12} \) and \( M_{32} \), we could obtain the flows shown in Figure 12.4.

All we have accomplished is to match \( M_{12} \), but at the expense of no longer matching \( M_{13} \). What we need is a procedure that uses the information available from all three meters to produce the best estimate of the actual angles, line flows, and bus load and generations.

Before proceeding, let's discuss what we have been doing. Since the only thing we know about the power system comes to us from the measurements, we must use the measurements to estimate system conditions. Recall that each instance the measurements were used to calculate the bus phase angles a bus 1 and 2. Once these phase angles were known, all unmeasured power flow loads, and generations could be determined. We call \( \theta_1 \) and \( \theta_2 \) the state variable for the three-bus system since knowing them allows all other quantities to be calculated. In general, the state variables for a power system consist of the bus voltage magnitude at all buses and the phase angles at all but one bus. The swing or reference bus phase angle is usually assumed to be zero radians. Not that we could use real and imaginary components of bus voltage if desired. If we can use measurements to estimate the "states" (i.e., voltage magnitudes and phase angles) of the power system, then we can go on to calculate any power flows, generation, loads, and so forth that we desire. This presumes that the network configuration (i.e., breaker and disconnect switch statuses) is known and that the impedances in the network are also known. Automatic load tap changing autotransformers or phase angle regulators are often included in a network, and their tap positions may be telemetered to the control as a measured quantity. Strictly speaking, the transformer taps and phase angle regulator positions should also be considered as states since they must be known in order to calculate the flows through the transformers and regulators.

To return to the three-bus DC power flow model, we have three meters providing us with a set of redundant readings with which to estimate the two states \( \theta_1 \) and \( \theta_2 \). We say that the readings are redundant since, as we saw earlier only two readings are necessary to calculate \( \theta_1 \) and \( \theta_2 \), the other reading is always "extra." However, the "extra" reading does carry useful information and ought not to be discarded summarily.
This simple example serves to introduce the subject of static-state estimation, which is the art of estimating the exact system state given a set of imperfect measurements made on the power system. We will digress at this point to develop the theoretical background for static-state estimation. We will return to our three-bus system in Section 12.3.4.

12.3 MAXIMUM LIKELIHOOD WEIGHTED LEAST-SQUARES ESTIMATION

12.3.1 Introduction

Statistical estimation refers to a procedure where one uses samples to calculate the value of one or more unknown parameters in a system. Since the samples (or measurements) are inexact, the estimate obtained for the unknown parameter is also inexact. This leads to the problem of how to formulate a "best" estimate of the unknown parameters given the available measurements.

The development of the notions of state estimation may proceed along several lines, depending on the statistical criterion selected. Of the many criteria that have been examined and used in various applications, the following three are perhaps the most commonly encountered.

1. The maximum likelihood criterion, where the objective is to maximize the probability that the estimate of the state variable, $\hat{x}$, is the true value of the state variable vector, $x$ (i.e., maximize $P(\hat{x} = x)$).
2. The weighted least-squares criterion, where the objective is to minimize the sum of the squares of the weighted deviations of the estimated measurements, $\hat{z}$, from the actual measurements, $z$.
3. The minimum variance criterion, where the objective is to minimize the expected value of the sum of the squares of the deviations of the estimated components of the state variable vector from the corresponding components of the true state variable vector.

When normally distributed, unbiased meter error distributions are assumed, each of these approaches results in identical estimators. This chapter will utilize the maximum likelihood approach because the method introduces the measurement error weighting matrix $[R]$ in a straightforward manner.

The maximum likelihood procedure asks the following question: "What is the probability (or likelihood) that I will get the measurements I have obtained?" This probability depends on the random error in the measuring device (transducer) as well as the unknown parameters to be estimated. Therefore, a reasonable procedure would be one that simply chose the estimate as the value that maximizes this probability. As we will see shortly, the maximum likelihood estimator assumes that we know the probability density function (PDF) of the random errors in the measurement. Other estimation schemes could also be used. The "least-squares" estimator does not require that we know the probability density function for the sample or measurement error. However, if we assume that the probability density function of sample $c$ measurement error is a normal (Gaussian) distribution, we will end up with the same estimation formula. We will proceed to develop our estimation formula using the maximum likelihood criterion assuming normal distribution for measurement errors. The result will be a "least-squares" or more precisely a "weighted least-squares" estimation formula, even though we will develop the formulation using the maximum likelihood criteria. We will illustrate this method with a simple electrical circuit and show how the maximum likelihood estimate can be made.

First, we introduce the concept of random measurement error. Note that we have dropped the term "sample" since the concept of a measurement is much more appropriate to our discussion. The measurements are assumed to be independent of anything else that is, the value obtained from the measurement device is close to the true value of the parameter being measured but differs by an unknown error. Mathematically, this can be modeled as follows.

Let $z_{\text{meas}}$ be the value of a measurement as received from a measurement device. Let $z_{\text{true}}$ be the true value of the quantity being measured. Finally, let $\eta$ be the random measurement error. We can then represent our measured value as

$$z_{\text{meas}} = z_{\text{true}} + \eta$$

(12.1)

The random number, $\eta$, serves to model the uncertainty in the measurement. If the measurement error is unbiased, the probability density function of $\eta$ is usually chosen as a normal distribution with zero mean. Note that other measurement probability density functions will also work in the maximum likelihood method as well. The probability density function of $\eta$ is

$$\text{PDF}(\eta) = \frac{1}{\sigma \sqrt{2\pi}} \exp(-\eta^2/2\sigma^2)$$

(12.2)

where $\sigma$ is called the standard deviation and $\sigma^2$ is called the variance of the random number. PDF$(\eta)$ describes the behavior of $\eta$. A plot of PDF$(\eta)$ is shown in Figure 12.5. Note that $\sigma$, the standard deviation, provides a way to make the seriousness of the random measurement error. If $\sigma$ is large, the measurement is relatively inaccurate (i.e., a poor-quality measurement device), whereas a small value of $\sigma$ denotes a small error spread (i.e., a higher-quality measurement device). The normal distribution is commonly used for modeling measurement errors since it is the distribution that will result when many factors contribute to the overall error.
12.3.2 Maximum Likelihood Concepts

The principle of maximum likelihood estimation is illustrated by using a simple DC circuit example as shown in Figure 12.6. In this example, we wish to estimate the value of the voltage source, $X^{\text{true}}$, using an ammeter with an error having a known standard deviation. The ammeter gives a reading of $z_1^{\text{meas}}$, which is equal to the sum of $z_1^{\text{true}}$ (the true current flowing in our circuit) and $\eta_1$ (the error present in the ammeter). Then we can write

$$z_1^{\text{meas}} = z_1^{\text{true}} + \eta_1$$

(12.3)

Since the mean value of $\eta_1$ is zero, we then know that the mean value of $z_1^{\text{meas}}$ is equal to $z_1^{\text{true}}$. This allows us to write a probability density function for $z_1^{\text{meas}}$ as

$$\text{PDF}(z_1^{\text{meas}}) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp \left[ -\frac{(z_1^{\text{meas}} - z_1^{\text{true}})^2}{2\sigma_1^2} \right]$$

(12.4)

where $\sigma_1$ is the standard deviation for the random error $\eta_1$. If we assume that the value of the resistance, $r_1$, in our circuit is known, then we can write

$$\text{PDF}(z_1^{\text{meas}}) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp \left[ -\frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)^2}{2\sigma_1^2} \right]$$

(12.5)

Coming back to our definition of a maximum likelihood estimator, we now wish to find an estimate of $X$ (called $X^{\text{est}}$) that maximizes the probability that the observed measurement $z_1^{\text{meas}}$ would occur. Since we have the probability density function of $z_1^{\text{meas}}$, we can write

$$\text{prob}(z_1^{\text{meas}}) = \int_{z_1^{\text{meas}}}^{\infty} \text{PDF}(z_1^{\text{meas}}) \, dz_1^{\text{meas}} \quad \text{as } dz_1^{\text{meas}} \to 0$$

$$= \text{PDF}(z_1^{\text{meas}}) \, dz_1^{\text{meas}}$$

(12.6)

The maximum likelihood procedure then requires that we maximize the value of $\text{prob}(z_1^{\text{meas}})$, which is a function of $X$. That is,

$$\max_X \text{prob}(z_1^{\text{meas}}) = \max_X \text{PDF}(z_1^{\text{meas}}) \, dz_1^{\text{meas}}$$

(12.7)

One convenient transformation that can be used at this point is to maximize the natural logarithm of $\text{PDF}(z_1^{\text{meas}})$ since maximizing the $\text{Ln}$ of $\text{PDF}(z_1^{\text{meas}})$ will also maximize $\text{PDF}(z_1^{\text{meas}})$. Then we wish to find

$$\max_X \text{Ln}[\text{PDF}(z_1^{\text{meas}})]$$

or

$$\max_X \left[ -\frac{1}{2} \frac{\text{Ln}(\sigma_1 \sqrt{2\pi}) - \frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)^2}{2\sigma_1^2}}{2\sigma_1^2} \right]$$

Since the first term is constant, it can be ignored. We can maximize the function in brackets by minimizing the second term since it has a negative coefficient, that is,

$$\max_X \left[ -\frac{1}{2} \frac{\text{Ln}(\sigma_1 \sqrt{2\pi}) - \frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)^2}{2\sigma_1^2}}{2\sigma_1^2} \right]$$

is the same as

$$\max_X \left[ \frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)^2}{2\sigma_1^2} \right]$$

(12.8)

$$\min_X \left[ \frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)^2}{2\sigma_1^2} \right]$$

The value of $X$ that minimizes the right-hand term is found by simply taking the first derivative and setting the result to zero:

$$\frac{d}{dx} \left[ \frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)^2}{2\sigma_1^2} \right] = -\frac{(z_1^{\text{meas}} - \frac{1}{r_1} X)}{r_1 \sigma_1^2} = 0$$

(12.9)
FIG. 12.12 Six-bus system with measurements.
for the bus-voltage magnitudes and phase angles given the measurements shown in Table 12.2. The procedure took three iterations with $x^0$ initially being set to 1.0 pu and 0 rad for the voltage magnitude and phase angle at each bus, respectively. At the beginning of each iteration, the sum of the measurement residuals, $J(x)$ (see Eq. 12.30), is calculated and displayed. At the end of each iteration, the maximum $\Delta|E|$ and the maximum $\Delta\theta$ are calculated and displayed. The iterative steps for the six-bus system used here produced the results given in Table 12.3.

The value of $J(x)$ at the end of the iterative procedure would be zero if all measurements were without error or if there were no redundancy in the measurements. When there are redundant measurements with errors, the value of $J(x)$ will not normally go to zero. Its value represents a measure of the overall
The formula developed in the last section for the weighted least-squares estimate is given in Eq. 12.23, which is repeated here.

\[ x^{\text{est}} = [(H)^T[R^{-1}][H]]^{-1}(H)^T[R^{-1}]z^{\text{meas}} \]

where

- \( x^{\text{est}} \) = vector of estimated state variables
- \([H]\) = measurement function coefficient matrix
- \([R]\) = measurement covariance matrix
- \(z^{\text{meas}}\) = vector containing the measured values themselves

For the three-bus problem we have

\[ x^{\text{est}} = \begin{bmatrix} \theta_1^{\text{est}} \\ \theta_2^{\text{est}} \end{bmatrix} \quad (12.27) \]

To derive the \([H]\) matrix, we need to write the measurements as a function of the state variables \(\theta_1\) and \(\theta_2\). These functions are written in per unit as

\[ M_{12} = f_{12} = \frac{1}{0.2} (\theta_1 - \theta_2) = 5\theta_1 - 5\theta_2 \]

\[ M_{13} = f_{13} = \frac{1}{0.4} (\theta_1 - \theta_3) = 2.5\theta_1 \]

\[ M_{32} = f_{32} = \frac{1}{0.25} (\theta_3 - \theta_2) = -4\theta_2 \quad (12.28) \]

The reference-bus phase angle, \(\theta_3\), is still assumed to be zero. Then

\[ [H] = \begin{bmatrix} 5 & -5 \\ 2.5 & 0 \\ 0 & -4 \end{bmatrix} \]