"The story is told that young King Solomon was given the choice between wealth and wisdom. When he chose wisdom, God was so pleased that he gave Solomon not only wisdom but wealth also. So it is with science."

– Arthur Holly Compton

The Complex Exponential

\[ s(t) = Ae^{-(\alpha+j\omega)t} = Ae^{-\alpha t}(\cos \omega t - j \sin \omega t) \]
The Complex Sinusoid

\[ s(t) = Ae^{-j\omega t} \]
\[ = A(\cos \omega t - j \sin \omega t) \]
\[
\omega = 2\pi f
\]

The complex sinusoid is “concentrated” in frequency.

The Dirac Impulse

The Dirac Impulse

\[ \delta(t) \]

Defined by its *sifting property*:

\[
\int_{-\infty}^{\infty} \delta(t - t_0)s(t)dt = s(t_0)
\]

The Dirac impulse is “concentrated” in time.
Time-frequency Profiles

Time-frequency profile for a sinusoid

Time-frequency profile for an impulse
An Important Relationship

\[
\int_{-\infty}^{\infty} e^{j2\pi ft} df = \delta(t)
\]

\[
\int_{-\infty}^{\infty} e^{j2\pi ft} dt = \delta(f)
\]

Fourier Analysis and Synthesis

\[
s(t) \xrightarrow{\mathcal{F}} S(f)
\]

Synthesis:

\[
s(t) = \int S(f)e^{j2\pi ft} df
\]

Analysis:

\[
S(f) = \int s(t)e^{-j2\pi ft} dt
\]
Complex Sinusoid

\[ s(t) = e^{j2\pi f_0 t} \]

\[
S(f) = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi ft} \, dt \\
= \int_{-\infty}^{\infty} e^{-j2\pi (f-f_0) t} \, dt \\
= \delta(f - f_0)
\]

\[ e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f - f_0) \]

Impulse

\[ s(t) = \delta(t - t_0) \]

\[
S(f) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi ft} \, dt \\
= e^{-j2\pi ft_0}
\]

\[ \delta(t - t_0) \xrightarrow{\mathcal{F}} e^{-j2\pi ft_0} \]
One-sided Exponential

\[ s(t) = e^{-\alpha t} u(t) \]

\[ S(f) = \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi ft} dt \]
\[ = \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t} dt \]
\[ = \left. \frac{-1}{\alpha + j2\pi f} e^{-(\alpha + j2\pi f)t} \right|_{0}^{\infty} \]
\[ = \frac{1}{\alpha + j2\pi f} \]

Notes:
i) \( \alpha > 0; \)
ii) \( u(t) \) is the unit-step function:

\[ u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \]
**Rectangle (rect) Function**

\[
s(t) = \begin{cases} 
  1 & |t| \leq 1/2 \\
  0 & \text{otherwise}
\end{cases}
\]

\[
S(f) = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt \\
= e^{-j2\pi ft} \bigg|_{-1/2}^{1/2} \\
= e^{j\pi f} - e^{-j\pi f} \\
= \frac{j2\pi f}{j2\pi f} \\
= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)
\]

**Rect  \xrightarrow{\mathcal{F}} \text{Sinc}**

**Rect  \xleftarrow{\mathcal{F}} \text{Sinc}**
Some Fourier Transform Properties

\[ s(t - t_0) \leftrightarrow S(f) e^{-j2\pi ft_0} \]

\[ s(at) \leftrightarrow \frac{1}{|a|} S(f/a) \]

\[ S(0) = \int_{-\infty}^{\infty} s(t) dt \]

\[ s(0) = \int_{-\infty}^{\infty} S(f) df \]

Rect \( \leftrightarrow \) Sinc (scaled)