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An almost four-approximation algorithm for maximum weight triangulation

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Abstract We consider the following *planar maximum weight triangulation* (MAT) problem: given a set of n points in the plane, find a triangulation such that the total length of edges in triangulation is *maximized*. We prove an $\Omega(\sqrt{n})$ lower bound on the approximation factor for several heuristics: maximum greedy triangulation, maximum greedy spanning tree triangulation and maximum spanning tree triangulation. We then propose the Spoke Triangulation algorithm, which approximates the maximum weight triangulation for points in general position within a factor of almost four in $O(n \log n)$ time. The proof is simpler than the previous work. We also prove that Spoke Triangulation approximates the maximum weight triangulation of a convex polygon within a factor of two.

Keywords Triangulation · Maximum weight triangulation · Spoke triangulation · Approximation algorithm · Approximation ratio

1 Introduction

Let P be a set of n points in the plane. A *triangulation* $\mathrm{T}(P)$ of P is a maximal set of non-intersecting straight-line segments connecting points in P. The *weight* $|\mathrm{T}(P)|$ of $\mathrm{T}(P)$ is the sum of the *Euclidean* lengths of edges in $\mathrm{T}(P)$. The *minimum weight triangulation* (MWT) is a triangulation of P with minimum weight. The *maximum weight triangulation* (MAT) is a triangulation of P with maximum weight. Computing the former is a well known problem in computational geometry and has been shown to be NP-hard recently (Mulzer and Rote 2006).

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Related work Recent research attention has been paid to solving the maximum weight triangulation problem as this may deepen our knowledge in studying optimal triangulations. The first work on it is in Wang et al. (1999) which presents an $O(n^2)$ algorithm to find the maximum weight triangulation of an n-sided polygon inscribed in a disk. Later, the conference version of this paper (Hu 2003) proposes the spoke triangulation which leads to a 6-approximation for maximum weight triangulation of the point set in general position. The approximation ratio is improved to 4.238 in Chin et al. (2004) through giving better weight bound on maximum weight triangulation and the ratio approaches four when n goes to infinity. In this paper, the approximation ratio is also proved to be $4 + \epsilon$, where $\epsilon = \frac{3.5}{\sqrt{n}}$ and the proof is simpler than Chin et al. (2004). When restricted to a convex polygon, a linear-time approximation scheme exists (Qian and Wang 2004).

In contrast to very limited research work in MAT, much more results are known on MWT. An important research direction on MWT is to compute approximations for the optimal triangulations. Levcopoulos and Krznaric (1998a) propose a constant factor approximation to MWT(P), and a $1 + \epsilon$ approximation for MWT of a convex polygon can be computed in linear time (Levcopoulos and Krznaric 1998b). Very recently, Remy and Steger show that a $1 + \epsilon$ approximation for a general point set can be computed in $n^{O(\log^8 n)}$ time (Remy and Steger 2006). This is interesting especially considering the NP-hardness nature of the MWT problem (Mulzer and Rote 2006). Other known heuristic algorithms are described in Plaisted and Hong (1987), Lingas (1987) and Heath and Pemmaraju (1994). For example, Lingas (1987) and Heath and Pemmaraju (1994) introduce so-called *minimum spanning tree triangulation* and *greedy spanning tree triangulation* heuristics. However, Levcopoulos and Krznaric (1996) show that these heuristics can have worst-case approximation ratios of $\Omega(n)$, $\Omega(\sqrt{n})$, respectively.

Summary of results In this paper, we prove that for MAT, the worst-case approximation lower bound of maximum greedy triangulation, maximum greedy spanning tree triangulation and maximum spanning tree triangulation heuristics (which are the natural analogues of their MWT counterparts) is $\Omega(\sqrt{n})$. We also obtain a tight bound of $\Theta(n)$ for degenerate cases. We then present the *Spoke Triangulation* and prove that it approximates the maximum weight triangulation for points in general position within a factor of almost 4 and can be computed in $O(n \log n)$ time. A similar result has been given in Chin et al. (2004), however, our proof is simpler compared to theirs. We also prove that Spoke Triangulation approximates the maximum weight triangulation of a convex polygon within a factor of 2.

Definition and notation The *greedy triangulation* GT(P) of P is obtained by repeatedly adding a longest possible edge that does not properly intersect any of the previously generated edges. The *greedy spanning tree triangulation* GSTT(P) of P is obtained as follows: start with a maximum spanning tree of the greedy triangulation, triangulate optimally each of the simple polygons bounded by this spanning tree and the convex hull of P. The *maximum spanning tree triangulation* MSTT(P) of P is obtained similarly to GSTT(P) except that it starts with the Euclidean maximum non-crossing spanning tree of P.

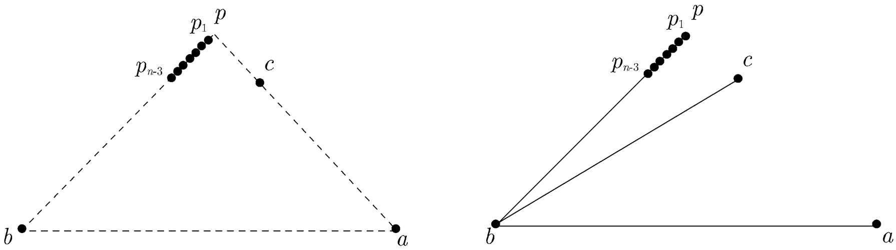


Fig. 1 (**a**) Degenerate case. (**b**) Maximum spanning tree for the point set

An algorithm is said to be a factor ρ approximation for a maximization problem if the algorithm is guaranteed to produce a solution whose objective function value is at least $(1/\rho)$ of the optimal solution.

Paper organization The rest of the paper is organized as follows: Sect. 2 shows the inapplicability of common heuristics to MAT. Section 3 describes the constant factor approximation algorithm to MAT. A summary of work is given in Sect. 4.

2 Lower bounds for common heuristics

This section will show that the worst-case approximation lower bound for MAT by maximum greedy triangulation (GT), maximum greedy spanning tree triangulation (GSTT) and maximum spanning tree triangulation heuristics (MSTT) is $\Omega(\sqrt{n})$. The approximation bound becomes tight, namely $\Theta(n)$, for degenerate cases.

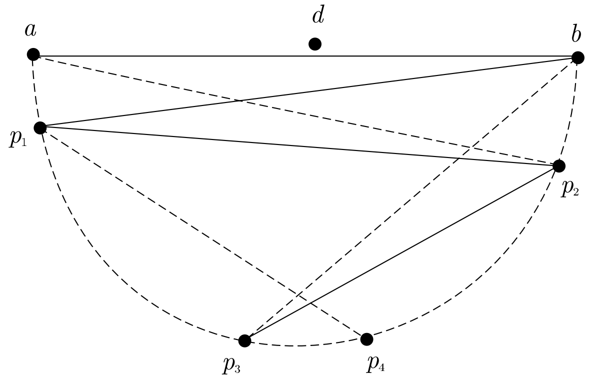
We first construct a point set P to prove the claim about GT. We start by allowing degeneracies, that is, we permit three or more collinear points in P.

Lemma 1 *For any integer $n \geq 0$, there exists a set P of n points in degenerate position such that* $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(n)$.

Proof Assume that the points in P are distributed as shown in Fig. 1(a). All the points are on the sides of the right triangle apb, where p is only a virtual point ($p \notin P$) satisfying $\angle apb = \pi/2$, $|ab| = n$, $|pb| = |pa| = n/\sqrt{2}$, $|pc| = 1$, and $|pp_j| < 1$ for $1 \leq j \leq n-3$.

By the definition of greedy triangulation, ab, bc, bp_{n-3} (recall that $p \notin P$) will be the first three edges to be added. After that, there is only one possible way to complete the triangulation: connect p_j to c for $1 \leq j \leq n-3$, connect p_j to p_{j+1} for $1 \leq j \leq n-4$ and connect a to c. Note that $|ab| + |bc| + |bp_{n-3}| + |ac| < 4n$ and $|p_j c| < \sqrt{2}$, $1 \leq j \leq n-3$, so $|\mathrm{GT}| = |ab| + |bc| + |bp_{n-3}| + \sum_{j=1}^{n-4} |p_i p_{i+1}| + \sum_{j=1}^{n-3} |p_i c| + |ac| = \Theta(n)$. On the other hand, a larger weight triangulation, denoted by $\mathrm{LWT}(P)$ (clearly $|\mathrm{LWT}(P)| \leq |\mathrm{MAT}(P)|$), is obtained by first connecting p_j to a for $1 \leq j \leq n-3$. Then $|\mathrm{LWT}| \geq \sum_{j=1}^{n-3} |p_j a| = \Theta(n^2)$, thus $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(n)$. This bound is tight immediately from Lemma 6. □

Fig. 2 Non-degenerate case



We next consider the non-degenerate case, that is, no three or more can be collinear. In Levcopoulos and Krznaric (1996) for proving similar claims for MWT, the authors start with the degenerate construction and then move the points p_j ($1 \le j \le n-3$) from a line to a curve (parabola) to prove the non-degenerate cases. However, this simple strategy will not work in our case. Our idea is that we distribute the first $\sqrt{n}$ points in a nice way on a unit half-circle such that the total length of the first $\sqrt{n}$ edges in GT(P) is $O(\sqrt{n})$. We then show that the total length of all remaining edges is also $O(\sqrt{n})$. However MAT(P) has the weight of $\Omega(n)$, which completes the proof.

Lemma 2 *For any integer $n \ge 0$, there exists a set P of n points in general position such that* $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(\sqrt{n})$.

Proof Refer to Fig. 2. Let ab be the diameter of the unit half-circle C and let d lie slightly above the center of C. For any two points x, y on C, let $\widehat{xy}$ denote the arc connecting x and y, so $|\widehat{ab}| = \pi$. All other points will be placed inside or on C. We say that p is *below* a chord xy, when p lies in, or on the boundary of, the closed region bounded by arc $\widehat{xy}$ and chord xy. Clearly, $|xy| > |xp|, |yp|$ (note that p lies in the closed half-disk). We distribute the next $k-2$ points, denoted by $p_1, p_2, \ldots, p_{k-2}$, on C as follows.

Place p_1 on C such that $|\widehat{bp_1}| = \pi/x_1$, where $1 < x_1 < 2$. Remaining points will be placed below bp_1. We want to let bp_1 be in GT(P) and this happens if all the remaining points lie below the chord p_1p_2 such that $|\widehat{bp_1}| > |\widehat{ap_2}|$. Let $x_2 = \frac{|\widehat{p_1b}|}{|\widehat{p_1p_2}|} > 1$ and restrict $x_2 < 2$, then $x_2 > \frac{1}{2-x_1}$. p_1p_2 will be in GT(P) if all other points lie below the chord p_2p_3 such that $|\widehat{p_1p_2}| > |\widehat{bp_3}|$. Let $x_3 = \frac{|\widehat{p_2p_1}|}{|\widehat{p_2p_3}|} > 1$ and restrict $x_3 < 2$, then $x_3 > \frac{1}{2-x_2}$.

We now show how to perform the above operations generally. Suppose that we have just added the edge $p_{j-1}p_j$, that is, the last placed point is p_{j+1}, and all the remaining points will lie below p_jp_{j+1}. Since $x_1, x_2, \ldots, x_{j+1}$ are determined and $x_{j+1} = \frac{|\widehat{p_jp_{j-1}}|}{|\widehat{p_jp_{j+1}}|}$, we have $|\widehat{p_{j-1}p_{j+1}}| = (1 - \frac{1}{x_{j+1}})|\widehat{p_jp_{j-1}}|$ and $|\widehat{p_jp_{j+1}}| = \pi/\prod_{k=1}^{j+1} x_k$.

To add edge $p_j p_{j+1}$, we need to place the point p_{j+2} on C such that all the remaining points lie below $p_{j+1}p_{j+2}$ to guarantee $|\widehat{p_j p_{j+1}}| > |\widehat{p_{j-1} p_{j+2}}|$. Let $x_{j+2} = \frac{|\widehat{p_{j+1}p_j}|}{|\widehat{p_{j+1}p_{j+2}}|}$, then $|\widehat{p_{j+1}p_{j+2}}| = |\widehat{p_j p_{j+1}}|/x_{j+2} = \pi/\prod_{k=1}^{j+2} x_k$. We have

$$
\begin{aligned}
&|\widehat{p_j p_{j+1}}| > |\widehat{p_{j-1} p_{j+2}}| && (1)\\
&\Rightarrow \quad |\widehat{p_j p_{j+1}}| > |\widehat{p_{j-1} p_{j+1}}| + |\widehat{p_{j+1} p_{j+2}}| && (2)\\
&\Rightarrow \quad |\widehat{p_j p_{j+1}}| > |\widehat{p_{j-1} p_{j+1}}| + \frac{|\widehat{p_j p_{j+1}}|}{x_{j+2}} && (3)\\
&\Rightarrow \quad \left(1 - \frac{1}{x_{j+2}}\right)|\widehat{p_j p_{j+1}}| > \left(1 - \frac{1}{x_{j+1}}\right)|\widehat{p_j p_{j-1}}| && (4)\\
&\Rightarrow \quad \left(1 - \frac{1}{x_{j+2}}\right)\frac{|\widehat{p_j p_{j-1}}|}{x_{j+1}} > \left(1 - \frac{1}{x_{j+1}}\right)|\widehat{p_j p_{j-1}}| && (5)\\
&\Rightarrow \quad x_{j+2} > \frac{1}{2 - x_{j+1}}. && (6)
\end{aligned}
$$

As one can see from the above process, the key to obtain the intended GT is to construct a sequence x_j such that $x_j = 1/(2 - x_{j-1}) + \epsilon$ ($\epsilon > 0$) and $1 < x_j < 2$. Noting that $x_j > 1/(2 - x_{j-1}) > x_{j-1}$, one can see that $x_1, x_2, \dots, x_{k-2}$ is a monotonic increasing sequence. This is why the points are not placed symmetrically on the circle C. Lemma 3 below can be used to construct this sequence. Suppose that we have added the points $p_1, p_2, \dots, p_{k-2}$ and thus determined $x_1, x_2, \dots, x_{k-2}$. Setting $x_1 = (k+1)/k$ and $\epsilon = 1/k^k$, the sequence is

$$
x_1 = \frac{k+1}{k}, \qquad x_j = \frac{1}{2 - x_{j-1}} + \frac{1}{k^k} \quad \text{for } j = 2, 3, \dots, k-2. \tag{7}
$$

Since any chord in C is shorter than its corresponding arc, we can safely use the length of the arc to bound the length of chord. Note that $1/\prod_{j=1}^{i} x_j < 1/x_1^i$ (since $x_1, x_2, \dots, x_j$ is a monotonic increasing sequence), then the total length of the first $k-2$ edges is

$$
\begin{aligned}
&|ab| + |bp_1| + \sum_{i=1}^{k-4} |p_i p_{i+1}| && (8)\\
&< \pi + \pi \sum_{i=1}^{k-4}\left(1\Big/\prod_{j=1}^{i} x_j\right) && (9)\\
&< \pi + \pi \sum_{i=1}^{k-4}(1/x_1^i) && (10)\\
&< \pi + \frac{\pi}{x_1 - 1} && (11)
\end{aligned}
$$

$$= O\left(\frac{1}{x_1 - 1}\right) = O(k). \tag{12}$$

According to Lemma 3, $\prod_{i=1}^{k-3} x_i > \frac{k+1}{k}\frac{k}{k-1}\cdots\frac{5}{4} = \frac{k+1}{4}$, then $|\widehat{p_{k-4}p_{k-3}}| = \pi/\prod_{i=1}^{k-3} x_i = O(1/k)$. We now place the remaining $n-k-1$ points, which are below $p_{k-3}p_{k-2}$. They are further restricted to be (arbitrarily) distributed in an obtuse triangle $p_{k-3}p_{k-2}p_{k-1}$ with $p_{k-3}p_{k-2}$ being the longest edge. We also require $p_i d$ for $k \le i \le n-3$ must intersect $p_{k-4}p_{k-3}$. Since $p_{k-4}p_{k-3}$ has already been added to GT(P), no point below $p_{k-4}p_{k-3}$ can be visible to any point above it. Since any edge we can add is bounded above by $|\widehat{p_{k-4}p_{k-3}}| = O(1/k)$, the length of any triangulation of the region bounded by $p_{k-4}p_{k-3}p_{k-2}p_{k-1}$ is bounded above by $O(1/k) \times \Theta(n-k) = O(n/k)$. In order to complete the triangulation, we also need to add the edges on convex hull whose length is bounded by π, then $|\mathrm{GT}| = O(k) + O(n/k) + \pi$. Choose $k = \sqrt{n}$, then $|\mathrm{GT}| = O(\sqrt{n})$.

A larger weight triangulation LWT(P) is easier to compute. We connect d to $p_1, p_2, \ldots, p_{k-2}$ and all other $n-k-1$ points ($p_{k-1}, p_{k+2}, \ldots, p_{n-3}$). Each of first $k-2$ edges is longer than 1/2 and we have the following observation on the remaining $n-k-1$ edges. Through d we make a line perpendicular to $p_{k-3}p_{k-2}$ intersecting $p_{k-3}p_{k-2}$ at f. This is valid because d is very close to the center of C. Since $|p_{k-3}p_{k-2}| = O(1/k)$, $|df| > \sqrt{1-1/k^2}$. Since each of the $n-k-1$ remaining edges is bounded below by $\sqrt{1-1/k^2}$, we have $|\mathrm{LWT}| > (k-2)/2 + (n-k-1)\sqrt{1-1/k^2} = \Omega(n)$ for $k = \sqrt{n}$ even if LWT is incomplete. It follows that $|\mathrm{MAT}|/|\mathrm{GT}| = \Omega(\sqrt{n})$. □

Lemma 3 *For sufficiently large k and $\epsilon = \frac{1}{k^k}$, if a sequence $x_1, x_2, \ldots, x_{k-2}$ is defined as $x_1 = (k+1)/k$, and $x_j = 1/(2 - x_{j-1}) + \epsilon$, for $j = 2, 3, \ldots, k-2$, then*

$$1 < \frac{k-(j-2)}{k-(j-1)} < x_j < \frac{k-(j-1)}{k-j-\epsilon k^j} + \epsilon \quad \text{for } j = 2, 3, \ldots, k-2.$$

Proof First note that $1 < k - j - \epsilon k^j$ for $j = 2, 3, \ldots, k-2$. We prove $x_j < \frac{k-(j-1)}{k-j-\epsilon k^j} + \epsilon$ by induction. For $j = 2$, it is easy to verify $x_2 < \frac{k-1}{k-2-\epsilon k^2} + \epsilon$. Assuming $x_j < \frac{k-(j-1)}{k-j-\epsilon k^j} + \epsilon$ holds for $j = i-1$, we show that it also holds for $j = i$ for sufficiently large k,

$$x_i \tag{13}$$

$$= \frac{1}{2 - x_{i-1}} + \epsilon \tag{14}$$

$$< \frac{1}{2 - (\frac{k-(i-2)}{k-(i-1)-\epsilon k^{i-1}} + \epsilon)} + \epsilon \tag{15}$$

$$= \frac{k-(i-1)-\epsilon k^{i-1}}{k-i-2\epsilon k^{i-1} - \epsilon(k-(i-1)-\epsilon k^{i-1})} + \epsilon \tag{16}$$

$$< \frac{k-(i-1)}{k-i-\epsilon k^i} + \epsilon. \tag{17}$$

We then prove $\frac{k-(j-2)}{k-(j-1)} < x_j$ also by induction. For $j = 2$, it is easy to verify $\frac{k}{k-1} < x_2$. Assuming that $\frac{k-(j-2)}{k-(j-1)} < x_j$ holds for $j = i - 1$, we show that it also holds for $j = i$,

$$\begin{aligned} & x_i && (18)\\ &= \frac{1}{2 - x_{i-1}} + \epsilon && (19)\\ &> \frac{1}{2 - x_{i-1}} && (20)\\ &> \frac{1}{2 - \frac{k-(i-3)}{k-(i-2)}} && (21)\\ &= \frac{k-(i-2)}{k-(i-1)}. && (22)\end{aligned}$$

□

From Lemmas 1 and 2, we conclude

Theorem 4 *For any integer $n \geq 0$, there exists a set P of n points in general position such that* $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(\sqrt{n})$.

We next construct examples to show that $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GSTT}(P)|} = \Omega(\sqrt{n})$ and $\frac{|\mathrm{MAT}(P)|}{|\mathrm{MSTT}(P)|} = \Omega(\sqrt{n})$. When degeneracies are allowed, we reuse the example of Fig. 1(a). The maximum spanning tree of $\mathrm{GT}(P)$ and maximum spanning tree of P are the same (refer to Fig. 1(b)), so $|\mathrm{GSTT}| = |\mathrm{GT}|$ and $|\mathrm{MSTT}| = |\mathrm{GT}|$ even if the optimal algorithm is used for completing the triangulation. In non-degenerate case, we reuse the example in Fig. 2. It is not hard to see that edges ab, bp_1, $p_j p_{j+1}$ for $j = 1, 2, \ldots, k-4$ also belong to $\mathrm{MSTT}(P)$ and $\mathrm{GSTT}(P)$, thus $|\mathrm{GSTT}(P)|$ and $|\mathrm{MSTT}(P)|$ are asymptotically the same as $|\mathrm{GT}(P)|$ since no point below $p_{k-4}p_{k-3}$ can be visible to any point above $p_{k-4}p_{k-3}$. Thus, the approximation lower bound will not change. Hence, we have

Theorem 5 *For any integer $n \geq 0$, there exists a set P of n points in general position such that* $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GSTT}(P)|} = \Omega(\sqrt{n})$, *and* $\frac{|\mathrm{MAT}(P)|}{|\mathrm{MSTT}(P)|} = \Omega(\sqrt{n})$.

3 A constant-factor approximation for MAT

3.1 An almost four-approximation algorithm

The *diameter* of P is the longest segment with both endpoints in P. For points in general position, we present a new triangulation algorithm, which approximates $\mathrm{MAT}(P)$ within a small constant factor. We call it *Spoke Triangulation* algorithm (ST in short). Refer to Fig. 3 and Algorithm 1.

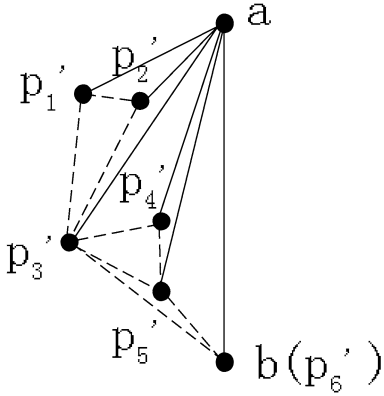


Fig. 3 Spoke triangulation (subset R)

Algorithm 1 Spoke triangulation

1: Compute the diameter ab of P. Put $D = |ab|$. Let $R = \{p_i \mid p_i$ lies to the left of the line $ab\}$ and $S = \{q_i \mid q_i$ lies to the right of the line $ab\}$. Let n_1 (resp. n_2) denote the cardinality of R (resp. S). Then $n_1 + n_2 = n - 2$. Since at most one of R and S can be empty, without loss of generality, we assume that R is not empty. Connect a to b.

2: At least one of $\sum_{i=1}^{n_1} |p_i a|$ and $\sum_{i=1}^{n_1} |p_i b|$ is larger than $n_1 D/2$ since $\sum_{i=1}^{n_1} |p_i a| + \sum_{i=1}^{n_1} |p_i b| > n_1 |ab|$ by triangle inequality. Without loss of generality, we assume $\sum_{i=1}^{n_1} |p_i a| > n_1 D/2$ and connect all p_i to a. Similarly, we connect all q_i to b assuming $\sum_{i=1}^{n_2} |q_i b| > n_2 D/2$.

3: We now have a spanning tree (called *Spoke Spanning Tree* or SST in short) of P and we complete the triangulation as in Graham's scan algorithm for computing convex hull. The algorithm for triangulating the region to the left of ab is given in lines 4–11. The region to the right of ab is triangulated similarly.

4: Sort $p_1, p_2, \ldots, p_{n_1}$ according to the angle $p_i ab$ (from largest to smallest angles), which results in a sequence $p'_1, p'_2, \ldots, p'_{n_1}$. Let p'_{n+1} denote b.

5: Add p'_1 in the list L_{left}.

6: **for** $i = 2$ **to** $n_1 + 1$ **do**

7:     Append p'_i to L_{left} and link p'_i to p'_{i-1}.

8:     **while** L_{left} contains more than two points **and** the last three points $(p'_j, p'_k, p'_l, l < k < j)$ in L_{left} make a left turn (i.e., $\angle p'_j p'_k p'_l > \pi$ in quadrilateral $p'_j p'_k p'_l a$). **do**

9:         Connect p'_j to p'_l and remove p'_k from the list.

10:     **end while**

11: **end for**

We are to prove the approximation bound of our algorithm. The first step is to give a bound for any MAT. Since any triangulation has at most $3n - 6$ edges, the obvious upper bound for the weight of any triangulation is $(3n - 6)D$ where D is the diameter of point set. Motivated by de Berg (2003), we here include the proof for a tighter bound on the weight of maximum weight triangulation.

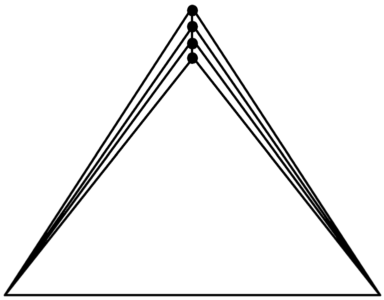


Fig. 4 Almost tight bound for any triangulation

Lemma 6 *For any triangulation* $\mathrm{T}(P)$ *of* P *with a diameter* D, $|\mathrm{T}(P)| < 2nD + 1.75\sqrt{n}D - 3.3D$.

Proof For every triangle $\triangle_i$ in the triangulation $\mathrm{T}(P)$, we have $\pi(\triangle_i) = h_i d_i/2$, where $\pi(\cdot)$ denotes the area, d_i denotes the length of the longest edge in $\triangle_i$, and h_i denotes the corresponding height (with respect to d_i) of $\triangle_i$. Clearly, $h_i < d_i$. Let $|\triangle_i|$ denote the weight (i.e., the perimeter) of $\triangle_i$. By triangular inequality, $|\triangle_i| < 2h_i + 2d_i$. Since $\sum_{\triangle_i \in \mathrm{T}(P)} \pi(\triangle_i) \le \pi D^2/4$ (Scott and Awyong 2000), $\sum_{\triangle_i \in \mathrm{T}(P)} h_i \cdot d_i \le \pi D^2/2$. Since $h_i < d_i$, we have $\sum_{\triangle_i \in \mathrm{T}(P)} h_i^2 < \pi D^2/2$. Denote by κ the number of triangles in $\mathrm{T}(P)$. Applying Claim 7, we have $\sum_{\triangle_i \in \mathrm{T}(P)} h_i < \frac{\sqrt{\pi\kappa}D}{\sqrt{2}}$.

$|\mathrm{T}(P)| = \frac{1}{2}\sum_{\triangle_i \in \mathrm{T}(P)} |\triangle_i| + \frac{1}{2}C_{conv}$ where C_{conv} denotes the perimeter of convex hull of P. Recall that the perimeter of the convex hull is bounded above by πD (Scott and Awyong 2000). We have

$$\begin{aligned} |\mathrm{T}(P)| &< \frac{1}{2}\sum_{\triangle_i \in \mathrm{T}(P)} \{2d_i + 2h_i\} + \frac{\pi D}{2} \\ &\le \sum_{\triangle_i \in \mathrm{T}(P)} \{D + h_i\} + \frac{\pi D}{2} < \sum_{\triangle_i \in \mathrm{T}(P)} \{D\} + \frac{\sqrt{\pi\kappa}D}{\sqrt{2}} + \frac{\pi D}{2}. \end{aligned} \tag{23}$$

Since we have at most $2n - 5$ triangles in $\mathrm{T}(P)$, it follows that $|\mathrm{T}(P)| < 2nD + 1.75\sqrt{n}D - 3.3D$. □

Note that the above upper bound is almost tight: if we place the interior $n - 3$ points very close to an vertex in an equilateral triangle, then the weight of MAT is almost $2nD$. Refer to Fig. 4. Such an example is also observed in Chin et al. (2004).

Claim 7 *Given* $h_i > 0, x > 0$, *and*

$$h_1^2 + h_2^2 + \cdots + h_n^2 < x^2,$$

one has

$$h_1 + h_2 + \cdots + h_n < x\sqrt{n}.$$

Proof We prove the claim by induction. It is easy to verify the fact when $n = 2$: since $(h_1 + h_2)^2 = h_1^2 + h_2^2 + 2h_1h_2 \le 2(h_1^2 + h_2^2) < 2x^2$, we have $h_1 + h_2 < \sqrt{2}x$.

Assuming that the claim holds for $n = j - 1$, we are to prove it is the case for $n = j$. Since $\sum_1^{j-1} h_i^2 < x^2$, $\sum_1^j h_i^2 < x^2 + h_j^2$. Thus, we need to show that $\sum_1^j h_i < \sqrt{x^2 + h_j^2}\sqrt{j}$. Noting that $\sum_1^{j-1} h_i < x\sqrt{j-1}$ or $\sum_1^j h_i < x\sqrt{j-1} + h_j$ by induction, it remains to show that $x\sqrt{j-1} + h_j \le \sqrt{x^2 + h_j^2}\sqrt{j}$, which can be easily verified. □

Theorem 8 *Spoke Triangulation algorithm properly triangulates a set of n points P (in general position) in $O(n \log n)$ time and $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < 4 + \epsilon$, where $\epsilon = \frac{3.5}{\sqrt{n}}$.*

Proof It is obvious that the algorithm properly triangulates P. We now bound the running time of $\mathrm{ST}(P)$. Step 1 requires $O(n \log n)$ time. Step 2 requires $O(n)$ time. Running time of step 3 (including steps 4–11) is bounded above by sorting $p_1, p_2, \ldots, p_{n_1}$ which takes $O(n \log n)$ time. Therefore, the total running time of $\mathrm{ST}(P)$ is $O(n \log n)$.

We are to bound the approximation ratio. $\mathrm{ST}(P)$ is composed of diameter, $p_i a$, $q_i b$ and the edges added in step 3. Suppose that there exists $p_t \in R$ such that $|p_t a| < D/2$ which means $|p_t b| > D/2$. An example point is p_2' in Fig. 3. According to ST, in the final triangulation there always exists a path from p_t to b containing no edge in $\{p_i a, q_i b, ab\}$. An example path is $p_2' p_3' b$ in Fig. 3. The length of the path is obviously longer than $|p_t b|$ which means $|\mathrm{ST}| > D/2 + D + \sum_{i=1}^{n_1} |p_i a| + \sum_{i=1}^{n_2} |q_i b| > 1.5D + (n_1 + n_2)D/2 = nD/2 + D/2$.

Otherwise, there is no such point, i.e., for all $p_i \in R$, $|p_i a| > D/2$. According to ST, there must exist an empty triangle abp_k in the final triangulation (for example, abp_5' in Fig. 3), then $|\mathrm{ST}| > D + |p_k a| + |p_k b| + \sum_{p_i \in R - \{p_k\}} |p_i a| + \sum_{i=1}^{n_2} |q_i b| > nD/2 + D/2$.

In either case, by Lemma 6 we have

$$\frac{|\mathrm{MAT}(P)|}{|\mathrm{ST}(P)|} < \frac{2nD + 1.75\sqrt{n}D - 3.3D}{nD/2 + D/2} < 4 + \epsilon, \tag{24}$$

where $\epsilon = \frac{3.5}{\sqrt{n}}$ and ϵ approaches zero as n grows. Note that a similar result has been given in Chin et al. (2004), but our proof is simpler. □

Observation 9 *For a convex polygon P, $|\mathrm{MAT}(P)| < nD + 0.15D$.*

Proof We have n edges on the convex hull and $n - 3$ edges in its interior. The perimeter of the convex hull is bounded above by πD according to Scott and Awyong (2000). Then $|\mathrm{MAT}(P)| < (n - 3)D + \pi D < nD + 0.15D$. □

Since $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < (nD + 0.15D)/(nD/2 + D/2) < 2$, $\mathrm{ST}(P)$ approximates $\mathrm{MAT}(P)$ of convex polygon P within a factor of two and can be computed in $O(n \log n)$ time compared with $O(n^3)$ time needed to compute MAT exactly by using dynamic programming. Thus we have shown

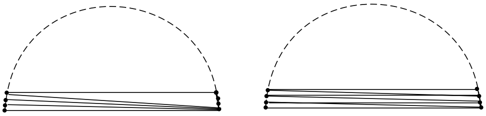


Fig. 5 Lower bound for Lemma 10: (**a**) ST(P) (**b**) MAT(P)

Theorem 10 *If P is a set of n points in convex position*, ST(P) *properly triangulates P in $O(n \log n)$ time and* $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < 2$.

Note that the above bound on approximation ratio is sharp since the example shown in Fig. 5 indicates that $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)|$ can be arbitrarily close to two. In the example, every point lies on a circle and two of them are the endpoints of the diameter. Half of the remaining points are near one endpoint of the diameter, and the other half are near the other endpoint.

4 Conclusion

This paper studies the planar maximum weight triangulation problem. We first prove that in the worst case maximum greedy triangulation, maximum greedy spanning tree triangulation and maximum spanning tree triangulation heuristics do not provide a constant factor approximation for the maximum weight triangulation. We then present the Spoke Triangulation whose length is always within a small constant factor from the maximum. Our proof is simpler compared to the previous work (Chin et al. 2004). The future work is to design a polynomial-time approximation scheme for MAT(P) of a general planar point set P. It is also interesting to give an NP-hardness proof for the maximum weight triangulation problem.

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