Want to reach:

\[ g_{th} = \frac{1}{2L} \ln \left[ \frac{1}{R_1 R_2 (1 - a_1)(1 - a_2)} \right] + \alpha \]

1. The small-signal gain, \( g_o \), must be \( > g_{th} \) for lasing
2. The introduction of mirrors introduces longitudinal (temporal) and transverse (spatial) modes on the beam
Longitudinal

Consider a Fabry-Perot resonator, i.e. two partially-reflecting mirrors:

Two partially-reflecting mirrors
Now $\cos 2\theta = b/a$, $\cos \theta = d/a$ so the extra path length is:

$$a + b = \frac{d}{\cos \theta} + \cos 2\theta \frac{d}{\cos \theta} = \frac{d}{\cos \theta} (1 + \cos 2\theta)$$

$$= \frac{d}{\cos \theta} (2 \cos^2 \theta) = 2d \cos \theta$$

So the phase difference is:

$$\phi = \frac{2\pi}{\lambda} (a + b) = \frac{4\pi}{\lambda} d \cos \theta$$

Then the total electric field is:

$$E = E_o t^2 + E_o t^2 r^2 e^{i\phi} + E_o t^2 r^4 e^{i2\phi} + \cdots$$

$$= E_o t^2 \sum_{n=0}^{\infty} r^{2n} e^{i n\phi} = \frac{E_o t^2}{1 - r^2 e^{i\phi}}$$
Next we allow for a phase shift at each reflection:

\[ r \rightarrow r = |r|e^{i\phi_r/2} \]

where:

\[ \phi_r/2 = \begin{cases} 
0 & \text{more dense to less} \\
\pi & \text{less dense to more} 
\end{cases} \]

Defining:

\[ R \equiv |r|^2 \]
\[ T \equiv |t|^2 \]

then:

\[ I_{\text{trans}} = I_o \frac{T^2}{|1 - Re^{i\Phi}|^2} \]

where: \( \Phi \equiv \phi + \phi_r \).
Then:

\[ I_{\text{trans}} = I_0 \frac{T^2}{(1 - Re^{i\Phi})(1 - Re^{-i\Phi})} \]

\[ = I_0 \frac{T^2}{1 - 2R\cos\Phi + R^2} \]

\[ = I_0 \frac{T^2}{(1 - R)^2 \left[ 1 + \frac{2R(1 - \cos\Phi)}{(1 - R^2)} \right]} \]

\[ = \frac{I_0 T^2}{(1 - R)^2 \left[ 1 + F' \sin^2(\Phi/2) \right]} \]

where:

\[ F' \equiv \frac{4R}{(1 - R)^2} \]

and [...] \equiv \text{Airy function.}
If $R + T = 1$, i.e. no losses due to absorption:

$$\frac{I_{\text{trans}}}{I_o} = \frac{1}{1 + F'\sin^2(\Phi/2)}$$

To find FWHM, set:

$$\frac{I_{\text{trans}}}{I_o} = \frac{1}{1 + F'\sin^2(\Phi/2)} = \frac{1}{2}$$

Then: $F'\sin^2(\Phi/2) = 1$ and for $\sin\Phi/2 \approx \Phi/2$,

$$\text{FWHM} = 4/\sqrt{F'}$$

Also define “finesse” $F \equiv \frac{\text{peak separation}}{\text{FWHM}}$ or:

$$F = \frac{2\pi}{4/\sqrt{F'}} = \frac{\pi\sqrt{R}}{1 - R}$$
For cavities with two different mirrors:

\[ F = \frac{\pi (R_1 R_2)^{1/4}}{1 - (R_1 R_2)^{1/2}} \]

Now assume more dense to less dense: \( \phi_r = 0 \). Then the peaks in Airy function occur at \( n = \frac{2d}{\lambda} \). The wavelength at max intensity is therefore: \( \lambda_n^{\text{max}} = \frac{2d}{n} \) or:

\[ \nu_n^{\text{max}} = n\left(\frac{c}{\eta}\right) \]

and successive peaks are at: \( \Delta \nu_{\text{sep}} = \nu_{n+1}^{\text{max}} - \nu_n^{\text{max}} = \frac{(c/\eta)}{2d} \), independent of \( n \).
If \( d \) is variable: Fabry-Perot interferometer

If \( d \) is fixed: Fabry-Perot etalon

E.G. for \( d=1\text{mm} \),

\[
\Delta \nu_{sep} = \frac{3 \times 10^8}{2 \times 10^{-3}} = 1.5 \times 10^{11} \text{Hz}
\]

or about 100 times typical Doppler linewidth. However, for a laser cavity of 25cm, \( \Delta \nu_{sep} = 0.6 \text{GHz} \).

Also, quality factor defined:

\[
Q \equiv \frac{\nu_0}{\text{FWHM}} = \frac{\nu_0 F}{\Delta \nu_{sep}}
\]
Airy Function

$R = 0.9$

$R = 0.4$

$R = 0.2$
Fabry-Perot cavity modes

Because the electric field must approximately go to zero at mirrors, longitudinal modes (standing waves) result. For He-Ne laser with $d=1\text{cm}$:

$$n = \frac{2d\nu}{c/\eta} = \frac{2d}{\lambda} \approx 31,000$$

half-cycles within cavity

Note that the surface of each mirror must be flat to $\sim \lambda/10$ to obtain high finesse (and $Q$).

Longitudinal laser modes

There is a build-up of energy in various modes once laser is turned on, assuming sufficient gain. Frequencies/wavelengths not supported by cavity are essentially damped out.
Requirements for development of longitudinal laser modes:

1. gain at a frequency exceeds losses per:

\[ g_{th} = \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2 (1 - a_1)(1 - a_2)} \right) + \alpha \]

2. there exists \( n \) such that \( \nu = \frac{nc}{2\eta L_t} \)

Consider He-Ne @ 632.8nm, \( L_t = 25cm \), \( \Delta \nu_{sep} = 0.6GHz \) and \( \Delta \nu_{Dopp} \approx 1.5GHz \). Then: \( \frac{\Delta \nu_{Dopp}}{\Delta \nu_{sep}} = 2.5 \Rightarrow 3 \text{ modes} \)

Illustration showing gain profile, Fabry-Perot, and product
**Transverse cavity modes**

For longitudinal modes, a plane wave was assumed but that is not quite right.

**Fresnel-Kirchhoff diffraction**

For a ray striking an aperture, should use Fresnel-Kirchhoff integral formula:

\[
U_P = \frac{-ik}{4\pi} \int \int_{\text{Area}} U_0 \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r} \left[ \cos(\vec{n}, \vec{r}) - \cos(\vec{n'}, \vec{r'}) \right] dA
\]
Consider 2 parallel mirrors:

\[ \cos \theta = \frac{d}{r}, \quad r = [d^2 + (x - x')^2 + (y - y')^2]^{1/2} \] and

\[ U'(x', y') \approx -\frac{ik}{4\pi} \int \int_{\text{Area}} U(x, y) \frac{e^{ikr}}{r} dx dy \]

Next, assume aperture ("object") is small relative to image, i.e. \( x, y \ll x', y' \). Then:
\[ r \approx d \left[ 1 + \frac{x'^2 + y'^2}{d^2} - 2 \frac{(x'x + y'y)}{d^2} \right]^{1/2} \approx \left[ d + \frac{(x'^2 + y'^2)}{2d} - \frac{(xx' + yy')}{d} \right] \]

And:

\[ U'(x', y') = C \int \int U(x, y) e^{-i \frac{k}{\alpha} (xx' + yy')} \, dx \, dy \]

Need solutions that pass back and forth between mirrors many times. That is, there can be a decrease in amplitude but transverse 'shape' is the same:

\[ U'(x', y') = \gamma U(x', y') = C \int \int U(x, y) e^{-i \frac{k}{\alpha} (xx' + yy')} \, dx \, dy \]

Therefore \( U(x, y) \) is its own Fourier transform.
Hermite Gaussian

\[ m = 5 \]
\[ m = 4 \]
\[ m = 3 \]