Instabilities (in diff. eqns.)

⇒ Identification.

Given an initial value problem involving a system of $S$ coupled, generally non-linear, first order O.D.E.'s

$$y_i' = f_i(x,y) \quad (1)$$

Let $y_i(x)$ denote the exact
solution which satisfies the initial conditions \( y(a) = y_0 \).

To determine if set is stiff, we probe the solutions in the neighborhood of the exact solutions, \( y(x) \):

\[
y' - J(x) \{ y - y(x) \} = f(x, y(x)) \tag{2}
\]

\( J(x) \equiv \) Jacobian matrix of partial derivatives \( \frac{df}{dy_i} \).
evaluated at \((x, y(x))\). Assume a small variation of \(J(x)\), the localized eigen solutions of (2) are \(e^{\lambda_i x}\), where \(\lambda_i \equiv \lambda_i(x)\) are the localized, non-degenerate, eigenvalues of \(J(x)\).

Thus, the solns. of (1) in the neighborhood of \(y(x)\) are the form:
\[ y = y(t) + \sum_{i=1}^{5} c_i \ e^{\lambda_i x} \phi_i \]

where \( c_i \)'s are constants, \( \phi_i \)'s are eigenvectors, and the transient eigensolutions decay with increasing \( x \) at rates proportional to:

\[ \frac{1}{\text{Re}(-\lambda_i)} \sim \tau_i \quad (\text{time-constant}) \]
Assuming local stability so that
\[ \text{Re}(\lambda_i) < 0 \quad i = 1, \ldots, 5, \]

Define: If:
\[
\begin{align*}
\frac{\max \text{Re}(-\lambda_i)}{\min \text{Re}(-\lambda_i)} &> 10 \\
\text{Max Re}(-\lambda_i) & \quad i = 1, 5 \\
\text{Min Re}(-\lambda_i) & \quad i = 1, 5
\end{align*}
\]

Then system of 5, 1st eqns, is "stiff."
E. G. Parallel RLC Circuit:

Characteristic eqn: (s-domain)
\[ s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \]

Roots:
\[ s_{1,2} = -\frac{1}{2RC} \pm \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}} \]

So,
\[ u = A e^{s_1 t} + A_2 e^{s_2 t} \]
Green's functions:
L. J. Gallagher & I. E. Perlman
"Lecture Notes in Mathematics,"
vol 362, pg 374.
\[ L_2 \equiv \frac{d}{dx} p(x) \frac{d}{dx} + r(x) \]

\[ \equiv \text{Sturm-Liouville self-adjoint operator} \]

\[ L_2 y(x) = f(x) \]

\[ y(x) = - \int_{a}^{b} G(x|1u) f(u) \, du \]

\[ G(x|1u) \equiv \text{Green's function, not unique, but depends on boundary conditions.} \]
Assume \( g(x_1) = g(x_2) = 0 \)

\[
\left( \frac{d}{dx} p(x) \frac{d}{dx} + r(x) \right) G(x|u) = -\delta(x-u)
\]

\[
\left( \frac{d}{dx} p(x) \frac{d}{dx} + r(x) \right) g(x) = 0 \tag{1}
\]

Define \( J(x) = p(x) \frac{d}{dx} g(x) \)

Then (1) becomes

\[
\frac{dg(x)}{dx} = \frac{J(x)}{p(x)}; \quad \frac{dJ(x)}{dx} = -r(x)g(x)
\]
There are two families of solutions:

\[ S_1(x) \text{ and } S_2(x) \]

which is regular at \( x_1, x_2, \) and \( g_1(x), g_2(x) = 0 \)

To make the solution unique,

\[ J_1(x) = p(x) \frac{d}{dx} S_2(x) \]

take

\[ J_2(x) = p(x) \frac{d}{dx} S_2(x) \]

Then

\[ \frac{d}{dx} \left( S_1(x) \right) = \frac{J_1(x)}{p(x)} \]

\[ \frac{d}{dx} \left( S_2(x) \right) = \frac{J_2(x)}{p(x)} \]

\[ J_1(x) = J_2(x) \]

\[ S_1(x), S_2(x) \]

\[ \frac{d}{dx} S_1(x) \]

\[ \frac{d}{dx} S_2(x) \]
\[
\frac{dS_2(x)}{dx} = \frac{J_2(x)}{P(x)} \\
\frac{dJ_2(x)}{dx} = -r(x) S_2(x)
\]

\[
\begin{align*}
S_2(x_2) &= 0 \\
J_2(x_2) &= 1
\end{align*}
\]

It is "straightforward" to show that
\[
J_1(x)S_2(x) - J_2(x)S_1(x) = A = \text{constant}
\]

\[
G(x \mid u) = \frac{1}{A} \begin{cases} 
  g_1(x)S_2(u), & x \leq u \\
  g_2(u)S_2(x), & x > u 
\end{cases}
\]
\[ y(x) = \frac{1}{A} \left[ \delta_2(x) \int_0^x \delta_1(u) f(u) du \right. \\
\left. + \ g_2(x) \int_0^x \delta_2(u) f(u) du \right] \]

Conditions for existence of Green's function are essentially the same as those for the existence of \( y(x) \).

Additionally, \( A \neq 0 \) 

\( \Rightarrow \) linearly independent of solutions.
Sufficient conditions for the existence of \( g_1 \) and \( g_2 \) are that \( r(x) \) and \( \frac{1}{p(x)} \) be piecewise continuous and finite on \( X, x_1 \leq x \leq x_2 \).